12. The scholarship problem

The theme of this unit is tracking financial system over time. We begin with a couple of examples of compound interest

Example 1. A sum of $100 is invested for a period of two years at an interest rate of 20% per year. There are three possible schemes for compounding the interest.

(a) For each scheme, calculate the amount in the account at each moment of the two-year period and plot a graph of this amount against time t ($0 \le t \le 2$).

(i) *Compounded annually.* At the end of each year, 20% of the minimum balance during the year is added to the account. Thus the amount of interest gained in the first year will itself gain interest during the second year.

(ii) *Compounded semi-annually*. At the end of each half-year, 10% of the minimum balance during that half-year is added to the account.

(iii) *Compounded quarterly*. At the end of each quarter-year, 5% of the minimum balance during the quarter-year is added to the account.

Notice what we doing with each option. We are cutting that 20% interest rate into n equal size pieces and also cutting each year into n equal periods. Then we are applying the interest rate pieces to each of the periods.

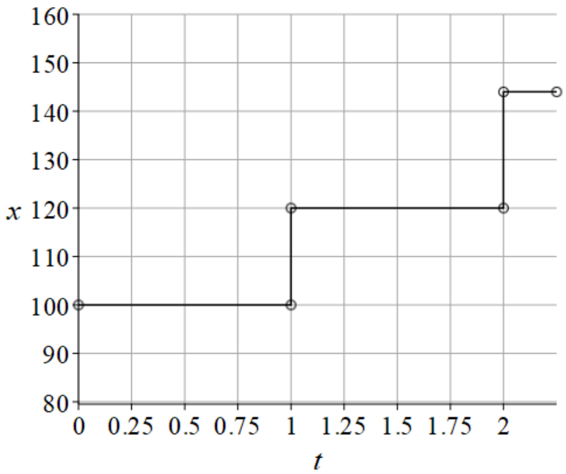
(b) The third of your three graphs has a sequence 8 vertical line segments. Show that the lengths of these line segments are in geometric sequence and find the multiplier.

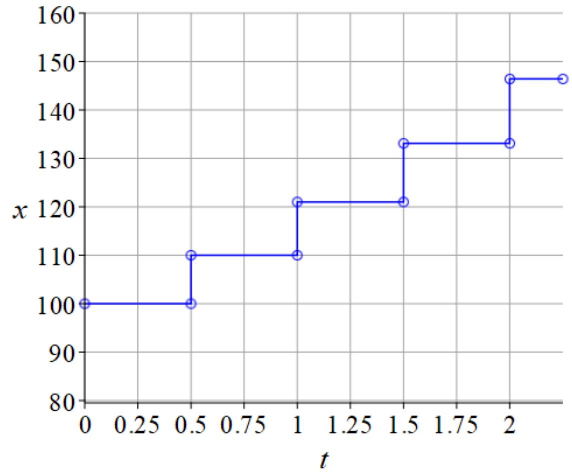
Solution
(a)
(i) Compounded annually. At the end of each year the account grows by 20%. That's a multiplier of 1.2 acting over 2 periods.

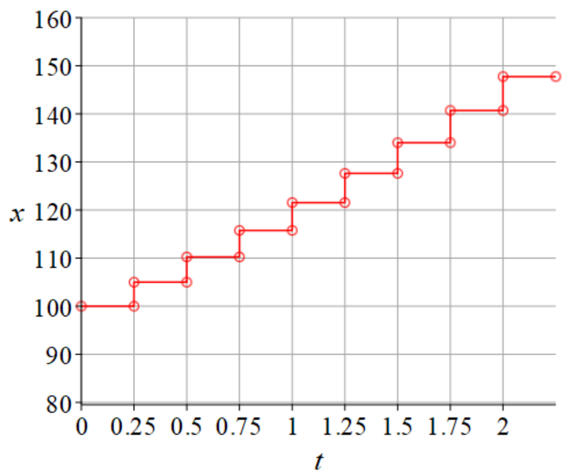
(ii) Compounded semi-annually. At the end of each half-year the account grows by 20%/2 = 10%. That's a multiplier of 1.1 acting over 4 periods.

(iii) Compounded quarterly. At the end of quarter-year the account grows by 20%/4 = 5%. That's a multiplier of 1.05 acting over 8 periods.

Annually r = 1.2	semi-annually r = 1.1	Quarterly r = 1.05
100.00 $100(1.2) = 120$ $100(1.2)^2 = 144$	100.00 $100(1.1) = 110$ $100(1.1)^2 = 121$ $100(1.1)^3 = 133.10$ $100(1.1)^4 = 146.41$	100.00 $100(1.05) = 105$ $100(1.05)^2 = 110.25$ $100(1.05)^3 = 115.76$ $100(1.05)^4 = 121.55$ $100(1.05)^5 = 127.63$ $100(1.05)^6 = 134.01$ $100(1.05)^7 = 140.71$ $100(1.05)^8 = 147.75$







(b) The lengths of the vertical segments of the third graph are the successive interest payments. These are all 5% of the amount at the start of the interval. Thus these are:

$$100.00(0.05)$$
$$100(1.05)(0.05)$$
$$100(1.05)^2(0.05)$$
$$100(1.05)^3(0.05)$$

Etc.

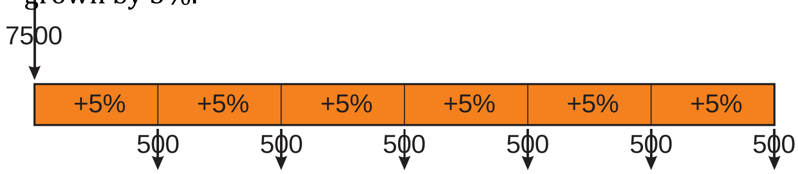
This clearly a geometric sequence with multiplier 1.05. This is expected as the amounts in the account are also growing geometrically with multiplier 1.05.

Example 2. **The scholarship problem**

The student council has decided to establish a scholarship of $500 which will be awarded each year to "the university-bound student who has contributed most to the life of the school." To finance this scholarship they canvass the alumni to donate to a trust fund.

This is a classic annuity problem. It can be analyzed in quite different ways. The standard "finance" analysis uses the concept of *present value* and we look at that here. But, following the ideas of the previous unit, we might also try to find an explicit solution to the recursive equation.

Suppose the fund is formally announced on a certain September 1, with a capital of $7500, with the first scholarship to be awarded next year. To be specific, over the course of the year the fund will gain interest at an annual rate of 5% and each year on September 1, $500 will be withdrawn from the fund for the winning student. Thus, the size of the fund rises and falls over the course of each year––each September 1 it falls by $500 and then by the start of next September the amount remaining has grown by 5%.



Here's the problem. If no additional capital is added to the fund, will it be enough to support the scholarship forever? If not, how long *will* the fund provide the scholarship? That is, how many $500 scholarships will it support?

Your job is to determine whether the fund will support the scholarship indefinitely, and if not, how many years it will last.

(a) *Some preliminary calculations.*

Let x_0 = 7500 be the initial amount in the fund and let x_n be the amount after *n* years. Let's just run the fund for a couple of years.

$$x_1 = (1.05)x_0 - 500 = (1.05)(7500) - 500 = 7375$$
$$x_2 = (1.05)x_1 - 500 = (1.05)(7375) - 500 = 7243.75$$
$$x_3 = (1.05)x_2 - 500 = (1.05)(7243.75) - 500 = 7105.94$$

Already most students could see that the fund would eventually run out. After one cycle (growth plus withdrawal) the fund is at 7375 and that's *less* than it was a year ago. The amount it grew during the year is less than the amount we took out at the end. Since now there is less in the fund than before, the amount it grows in the following year will be even less than before––but the amount we take out will be the same. Thus the fund will decrease by even more and clearly this will keep on getting worse and the fund will eventually run out.

Okay, then how many scholarships will we get?

Well we could answer that if we had a general expression for x_n in terms of *n*. We could then set it to zero and find the *n* at which x_n became zero.

Interestingly, this is *not* the way we are going to tackle this problem. Read on!

Analyzing the x-recursion.
Let's write down the recursive formula for our scholarship equation. The size x_n of the fund changes each year in two ways––first the amount from the previous year is increased by the 5% interest, and then it is decreased by the amount of the withdrawal:

$$x_{n+1} = (1.05)x_n - 500 \qquad x_0 = 7500$$

This equation has the general form:

$$x_{n+1} = rx_n - d \qquad x_0 = A$$

We encountered this equation in the previous section. There are a number of ways to play with it and find a general formula for x_n. Such a formula would of course allow us to decide exactly when the fund will run out of money.

This recursive equation combines additive and multiplicative change. We worked with a particular case of this in the last section.

Present value
But here we will follow another approach––in a sense we will start at the other end of the problem. Suppose we know exactly how many scholarships we want to endow with the donation. How much money will we have to raise to accomplish that?

To give ourselves a particular target, suppose we want the initial fund to provide the $500 scholarship for exactly 10 years.

Again we will put the money in the fund on Sept 1, and withdraw the first scholarship one year later. What should be the initial size A of the fund?

Here's the idea. The initial fund has to do 10 different jobs: fund the first scholarship, fund the second, fund the third… all the way to the 10th. So we can imagine the initial amount A as composed of 10 pieces each of which is just enough to fund one of the 10 scholarships:

$$A = a_1 + a_2 + a_3 + \cdots + a_{10}$$

Thus a_1 will fund the first (one year later), a_2 will fund the second (two years later) etc. We will figure out what each of the a_k have to be. And that will give us A.

Let me first ask, will the a_i all be the same? The answer is no. Each of the a_i will have some time to grow (at interest rate i = 0.05) but the later ones can be smaller because they will have more time to grow (and they all have the same target: d = 500).

Let's calculate the a_i. The first will have one year to grow and in that time it will grow to $a_1 r$ where $r = 1+i = 1.05$ is the one-year multiplier. Since that must give us d, we have

$$a_1 r = d.$$

Second, a_2 will have two years to produce d, so

$$a_2 r^2 = d.$$

This is in fact the general pattern:

$$a_k r^k = d$$

We can solve these equations for the a_k:

$$a_k = \frac{d}{r^k}.$$

These solve to give:

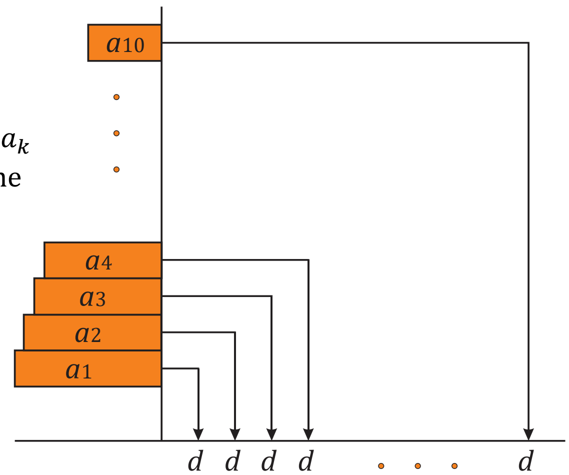
$$a_1 = \frac{d}{r} = \frac{500}{1.05} = 476.19$$

$$a_2 = \frac{d}{r^2} = \frac{500}{1.05^2} = 453.51$$

$$a_3 = \frac{d}{r^3} = \frac{500}{1.05^3} = 431.92$$

Etc,

The set-up is shown in the diagram at the right. Each of the a_k has a different number of years to grow to the size of d, so the later ones can be smaller.



I find this a wonderful formula. It has a simple elegant structure that makes perfect sense to me.

That gives us a formula for the initial investment A.

$$A = a_1 + a_2 + a_3 + \cdots + a_{10}$$

$$= \frac{d}{r} + \frac{d}{r^2} + \frac{d}{r^3} + \cdots + \frac{d}{r^9} + \frac{d}{r^{10}}$$

There is some standard terminology here. A is called the *present value* of the sequence of 10 scholarships. It's the amount of money you need to have *now* to fund a series of future costs.

Recall that our problem was this: given the size of the scholarship (d), the multiplier $r = 1 + i$) and the number of scholarships (10), find the initial investment A. Well we have done that—we have a formula for A in terms of d and r and 10. I rewrite it as:

$$A = \frac{d}{r}\left(1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots + \frac{1}{r^9}\right)$$

The sum in the brackets is a geometric series with common ratio $1/r$. We know how to find its sum.

$$A = \frac{d}{r}\left(\frac{1 - (1/r^{10})}{1 - (1/r)}\right)$$

Simplify

$$A = d\left(\frac{1 - (1/r^{10})}{r - 1}\right)$$

Here we have $r = 1.05$ and $d = 500$. That gives us $A = \$3860.87$

This seems reasonable. If the interest rate was zero, we'd need

$A = 10 \times 500 = 5000$.

So the interest has saved us more than \$1100.

Back to the beginning
Our original problem was the question of how many scholarships the endowment of $7500 would support?

If we let that number be *n* (instead of 10), the equation above becomes:

$$A = d\left(\frac{1-(1/r^n)}{r-1}\right)$$

and we know everything except *n*. Rearranging:

$$\frac{A}{d}(\mathrm{r}-1) = 1-(1/r^n)$$

$$\frac{1}{r^n} = 1-\frac{A}{d}(r-1)$$

$$\frac{1}{1.05^n} = 1-\frac{7500}{500}(0.05) = \frac{1}{4}$$

$$1.05^n = 4$$

Trying different values of *n* we find:

$$1.05^n = 4$$

$$1.05^{28} = 3.920$$

$$1.05^{29} = 4.116$$

So we will get 28 scholarships, but will not be able to fund the 29th.

The formal way to solve this equation is to use logarithms, but you might not meet these until grade 12. For this problem, it is easy enough to use guess and check.

Here's an interesting question. Immediately after we fund the 28th scholarship, how much will remain in the fund?

This is an excellent problem to throw out to the class.

To answer this, let's see how much we will need to fund the 28 scholarships. That will be

$$A = d\left(\frac{1-(1/r^{28})}{r-1}\right)$$

$$= 500\left(\frac{1-(1/1.05^{28})}{0.05}\right) = 7449.06$$

That leaves us with

$$7500 - 7449.06 = \$50.94$$

Of course over the next year it will gain some interest and will grow a bit, but it will certainly not be enough to fund another scholarship.

Problems

1.(a) You have \$10,000 which is targeted to support your first year at university. You decide to put it in a savings account for two years. But there are several different interest schemes available.
(i) 10% interest compounded annually. The funds grow by 10% per year. At the end of each year, 10% of the minimum balance during the year is added to the account.
(ii) 9.6% compounded monthly. At the end of each month, $\frac{9.6}{12}$% of the minimum balance during the month is added to the account.
(iii) 9.5% compounded daily. At the end of each day, $\frac{9.5}{365}$% of the minimum balance during the day is added to the account. We are assuming that both years have 365 days.
(iv) 9.4% compounded hourly. At the end of each hour, $\frac{9.4}{365\times 24}$% of the minimum balance during the hour is added to the account.

Calculate the amount of money you will have at the end of the two years under each scheme. Which is the best one to take?

2. (a) A sum of \$10,000 is invested for a period of one year at an annual interest rate of 6% under four different compounding schemes. For each scheme, calculate the amount in the account at each moment of that period
(i) simple interest.
(ii) compounded monthly.
(iii) compounded daily.
(iv) compounded hourly.

(b) Suppose you keep your money in the account for 10 years with the same interest rates. Calculate the amount at the end for each of the options in (a). Pretend there are no leap years so that every year is 365 days.

3. The recursive equation for the size of the scholarship fund can be written:

$$x_{n+1} = rx_n - d \qquad x_0 = A$$

where A is the initial fund, d is the value of the scholarship and $r = 1 + i$ is the annual multiplier (so that i is the interest rate) It has general solution:

$$x_n = r^n A - \left(\frac{r^n - 1}{r - 1}\right) d$$

It turns out that there are three things that can happen to x_n as n increases.
(a) it can increase indefinitely
(b) it can stay constant
(c) it can decrease and eventually become zero.

In terms of the three parameters A, d and i, find the conditions that characterize each case. Interpret this condition and explain why it makes intuitive sense. We will assume that all three of these quantities are positive.

4. I want to endow a series of scholarships over the 10 years. I will use the set-up of this section: the amount in the fund will grow at an annual interest rate of 5% (compounded annually) and the scholarships are each valued at $500. However, at the end of the first year (one year after the initial endowment) I want one scholarship to be awarded; after two years I want two scholarships to be awarded; after three years I want three scholarships etc. and at the end, after ten years, I want ten scholarship. How much should the endowment be?

5. I have an account at the local Trust Company which pays 0.1% interest every week. To be precise, every Friday at 6 PM an amount is deposited into the account equal to 0.1% of the minimum balance maintained during the previous week (since last Friday at 6 PM). I use the account to buy my morning coffee each day. Specifically, every day (weekends and holidays included) at 8 AM, I withdraw exactly $1 from the account and use it to buy a cup of coffee. Suppose on a certain Friday at noon, the account holds exactly $5000. The problem is to determine whether the account will support my coffee-drinking habit indefinitely.
(a) Tabulate the balance in the account over the next few weeks, always making your calculation on Friday at noon.
(b) Argue that the account will eventually run out.
(c) By any method, find a formula for the balance in the account after *n* weeks, and determine when I will have to jettison my coffee habit. [Answer: after 336 weeks.]

The "change of variable" solution to the general linear recursion.
6. In this section we solved the recursive equation

$$x_{n+1} = rx_n - d \qquad x_0 = A$$

by iterating it starting at $x_0 = A$ and seeing a pattern. Here we look at a direct method to solve the equation.

(a) Start with the particular equation:

$$x_{n+1} = (1.05)x_n - 500 \qquad x_0 = 7500$$

Now change variable with the substitution

$$z_n = 10000 - x_n$$

That is, use this change of variable equation to replace all occurrences of *x* by *z* in the recursion. The new *z*-recursion has a simple structure which allows you to solve it easily. Now return to the variable *x* (change from *z* to *x*) and you will get the correct formula. This is a cool method for solving equations of the above type.

(b) Where did I get the 10000 from in the change of variable formula? What was it that made this substitution work? Analyze it and develop a change-of-variable method to solve the general recursion

$$x_{n+1} = rx_n - d \qquad x_0 = A.$$

(b) Where did I get the 10000 from in the change of variable formula? What was it that made this substitution work? Analyze it and develop a change-of-variable method to solve the general recursion

$$x_{n+1} = rx_n - d \qquad x_0 = A.$$

7. The general linear recursion

$$x_{n+1} = rx_n - d \qquad x_0 = A$$

has two special forms:

(a) Arithmetic (r = 1): $x_{n+1} = x_n - d \qquad x_0 = A$

(b) Geometric (d = 0): $x_{n+1} = rx_n \qquad x_0 = A$

Show that the general solution

$$x_n = r^n A - \left(\frac{r^n - 1}{r - 1}\right) d$$

gives the correct solution in the two special cases.