2. Counting Trains

This is quite an interesting activity. In line with my big theme of parallel structures, it presents the students with two worlds:

Fibonacci numbers and Pascal's triangle both appear in the Grade 11U curriculum but my experience is that most teachers do very little with them.

That's a missed opportunity as they are a great resource for the development of mathematical thinking.

The Fibonacci world. A sequence of integers generated with one of the simplest of all recursive generators: $x_{n+1} = x_n + x_{n-1}$. This is an abstract world which contains many unexpected patterns, some of which we will encounter here.

The trains world. A sequence of integers with a concrete realization (number of trains of a fixed length) giving us a powerful access to the surprising patterns of the Fibonacci world.

In this unit the students are challenged to provide proofs for some of the well-known Fibonacci formulae by invoking properties of the structure of trains.

Problem 1. I want to construct a train of total length n units using cars which are either 1 unit long or 2 units long. The question is, for each value of n, how many different trains are there?

1

2

For example, for trains of length 12, here is one possibility, using 6 cars of length 1 and 3 cars of length 2.

1	**1**	**1**	**2**	**2**	**1**	**2**	**1**	**1**

Other possibilities are obtained by rearranging the cars, or using different numbers of short and long cars. Note that a train has a front and a back, so that the mirror image of the train we have just seen:

1	**1**	**2**	**1**	**2**	**2**	**1**	**1**	**1**

is counted as a different train.

Of course for small n we can simply write all the possible trains down. Thus the table at the right shows that there are 8 different trains of length 5. To have some notation, we let t_n denote the number of trains of length n. Thus we have shown that $t_5 = 8$.

For example, here are the 8 trains of length 5
1-1-1-1-1
1-1-1-2
1-1-2-1
1-2-1-1
2-1-1-1
1-2-2
2-1-2
2-2-1

How do we handle large *n*? There is in fact a standard combinatorial approach,--looking at the different possibilities for the number of 2-cars, and counting the number of arrangements of each. But our objective here is to make use of the power of recursive thinking.

We start by collecting some data––actually counting the different trains for small values of *n* by listing the possibilities. The numbers that we get turn out to be the beginning of the well-known Fibonacci sequence. The "law" of these numbers is that each term is the sum of the two preceding terms.

Length of train n	Number of trains t_n
1	1
2	2
3	3
4	5
5	8
6	13

For many students, once they see this pattern, they figure that the problem is solved, because we can continue this pattern and get the number of trains of any length. For example, the number of length 7 will be 8+13=21, etc. But can we be sure of that? Do we know for sure that the pattern continues? This is our first problem.

A question of elegance.

Let me point out that there's more here than a question of certainty. If this really does hold, one would think there ought to be a simple elegant argument for it (after all there's nothing very complex going on here) and we are curious to discover what this might be.

Establishing the recursion.

Our task is to convince ourselves that the simple sum rule should hold for the train sequence. As a specific example, take the equation:

$$t_7 = t_6 + t_5$$

Find an argument that the number of 7-trains *has* to be the sum of the number of 6-trains and the number of 5-trains.

Length of train n	*Number of trains* t_n
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34

I give this question to the class, but they've never seen anything like it before and hardly know where to begin. How could such an argument ever be constructed?

I give the students a hint––well it's more than a hint, it's a simple but powerful approach that will serve them well in all the remaining problems we will look at.

The equation above asks you to show that one quantity (t_7) is the sum of two other quantities (t_6 and t_5). Now all three numbers are the sizes of concrete sets. Maybe there's a natural way of partitioning the 21 objects in the t_7-set into two types that corresponding readily to the objects in the two other sets (t_6 and t_5).

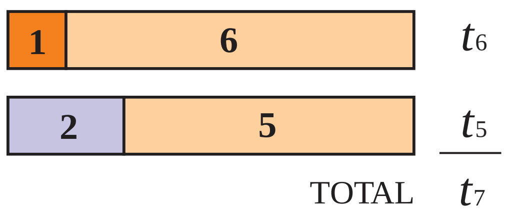
We are looking here for an argument that is "organic" in the sense that it is based naturally on the structure of trains.

For example imagine that you are atop the CN Tower looking down at all 21 trains of length 7, each of which has an engineer. Can you think of an instruction you can give the engineers: “if your train has the following property drive it to the east, and if it doesn’t, drive it to the west” such that there is a natural 1-1 correspondence between the trains that go east and all the trains of length 6, and between the trains that go west and all the trains of length 5.

This is the hint I give the students but most of them had more trouble with it than I expected.

Well it’s a type of thinking they might have never seen. That convinces me more than ever that this is the sort of analysis that needs to appear earlier in their lives.

Here’s the argument. The trains of length 7 are of two kinds: those that begin with a 1-car and those that begin with a 2-car. Now how large is each set? Clearly there are t_6 trains in the first (the rest of the train can be any train of length 6) and t_5 in the second (the rest of the train has length 5). So t_7 must be the sum of those two numbers.



Here’s the instruction for the engineer. “If your train starts with a 1-car, go east, and if it starts with a 2-car, go west.”

It’s clear that this argument is quite general, and could be used to show that the number of 8-trains is equal to the sum of the number of 7-trains and the number of 6-trains etc. So the additive rule always holds.

$$t_n = t_{n-1} + t_{n-2}$$

We can deduce from this that the train numbers are given by the famous Fibonacci numbers. Indeed the two sets of numbers start out the same (1, 1), and have the same generating law, so they have to remain the same.

Note that in the table at the right I started the count at *n*=0. Sometimes that’s convenient. I chose $t_0 = 1$ because that’s what I need to make the sum rule work at the first step. Can we make "train" sense of *n*=0? Perhaps we can––I guess there's only one train of length 0 and that's the train with no cars.

n	t_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

n	t_n	t_n^2	
0	1	1	2
1	1	1	5
2	2	4	13
3	3	9	34
4	5	25	89
5	8	64	
6	13	169	
7	21		
8	34		
9	55		
10	89		

Sums of squares. The Fibonacci numbers possess many wonderful arithmetic properties, but some of these are not so easy to prove.

Here's a spectacular idea. What we have constructed above is a physical "model" of the Fibonacci numbers. Let's try to use *that* to obtain a "train-theoretic" proof of arithmetic relationships among these numbers. Here's an example.

Problem 2. Take two consecutive Fibonacci numbers and add their squares. We always get a Fibonacci number. For example:

$$5^2 + 8^2 = 89$$

Writing this in trains-notation we have

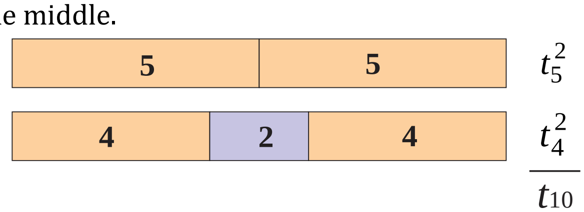
$$t_4{}^2 + t_5{}^2 = t_{10}$$

Can we find a "trains" argument that this must be true? This suggests that we look for a natural partition of the 10-trains into two disjoint classes with $t_5{}^2$ trains in the first class and $t_4{}^2$ trains in the second. A clue comes from noting that the subscripts 5 and 4 have the status of being roughly half of 10––this suggests that the classification ought to be based on something like "cutting the 10-trains in half." *Can you take it from there?*

This is a hard one. I ask the students: where have you seen squares arise in counting? There is silence.

How many outcomes of a die?
6.
How many of 2 different dice?
36.
And that's 6^2.

Well, here it is. There are two kinds of 10-trains, depending on whether or not they can be "cut in half." Those that can must have a join in the middle, and those that cannot must have a 2-car in the middle.



Those with a join are constructed of two trains of length 5, and there are t_5 possibilities for the front part and another possibilities for the back, for a total of t_5^2 trains with a join in the middle. Those with a 2-car in the middle are completed by adding two trains of length 4, and there are t_4^2 possibilities for that. Adding these up, we get

$$t_{10} = t_5{}^2 + t_4{}^2$$

Now that's an argument of great beauty!

What we've done here is quite remarkable. We have a surprising abstract mathematical identity, and so sense of why is might be true or how to prove it.

And then along comes the trains model for the numbers and suddenly we not only have a proof, but even an intuition for the relationship.

Problems

1. Study the table at the right. Column 2 displays the train numbers t_n. The sequence x_n in column 3 uses the same sum rule as the train numbers, but starts off in a different way. This sequence is completely specified by the generating recursion and the initial conditions.

$$x_{n+1} = x_n + x_{n-1} \qquad \begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$$

There are a number of simple general formulas for the terms of the *x*-sequence in terms of the train numbers. Find some of these, the simpler and the more elegant the better. Give three versions of your formula:

First in general (in terms of *n*): $x_n =$
Second for the case *n* = 7: $x_7 =$
Third using numerical values $29 =$

n	t_n	x_n
1	1	1
2	2	3
3	3	4
4	5	7
5	8	11
6	13	18
7	21	29
8	34	47
9	55	76
10	89	123

2. The sequence x_n of problem 1 is tabulated in column 2 at the right. It follows the rule

$$x_{n+1} = x_n + x_{n-1} \qquad \begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$$

Let S_n be the sum of the first *n* numbers of the *x*-sequence:

$S_1 = 1$
$S_2 = 1 + 3$
$S_3 = 1 + 3 + 4$
$S_4 = 1 + 3 + 4 + 7$

(a) Calculate the sums and fill in the entries of the S_n-column.

(b) There is an unexpected relationship between the entries of the S_n-column and the entries of the x_n-column. Find this relationship, give a few particular examples, and if you can, find a formal statement of the pattern (in terms of the general variable *n*).

n	x_n	S_n
1	1	
2	3	
3	4	
4	7	
5	11	
6	18	
7	29	
8	47	
9	76	
10	123	

3. [4 marks] The sequence x_n tabulated at the right is generated by the recursion

$$x_{n+1} = x_n + 2x_{n-1}$$

with initial conditions $x_1 = 1, x_2 = 1$. Let S_n be the sum of the first n terms:

$$S_n = x_1 + x_2 + x_3 + \cdots + x_n$$

(a) In the table at the right, enter the sums S_n for $1 \le n \le 9$.

(b) There is a simple relationship between the sums S_n and members of the sequence $\{x_k\}$. Actually there are many such relationships. Find a simple one and provide a precise statement of it. You might want to use a different formula for n odd and n even.

(c) Suppose I tell you that $x_{24} = 5592405$. Use your answer for (b) to find S_{24} and S_{25}
Note: You have to show your work. It's not good enough to use the amazing power of your calculator to generate the sequences up to n = 25 and read the answers off.

n	x_n	S_n
1	1	
2	1	
3	3	
4	5	
5	11	
6	21	
7	43	
8	85	
9	171	

4. [6 marks] Let s_n be the number of sequences of length n made up of 1's and 2's that have no consecutive 2's. We will call such a sequence "good." For example, of the following sequences of length 9, the first two are good and the second two are not good.

121111212
212121212
122111211
222211111

(a) By direct enumeration find the first 4 values of s_n and fill out the table at the right.

(b)Making use of the pattern that you find in the table, provide a conjecture for a recursive formula that might generate the s_n (that is, find a formula for s_n in terms of some of the preceding s_k).

(c) Provide a careful argument for your conjecture. Note that you are not asked to *solve* the recursion, but to make an argument, based on the property of the sequences, that your recursive formula must hold. If you don't want to work with a general n, you can do what we often do in class and work with particular numbers.

n	s_n
1	
2	
3	
4	

n	t_n
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

5. Take three successive train numbers, say, 3, 5, 8, and note that:

$$3\times5 + 5\times8 = 55$$

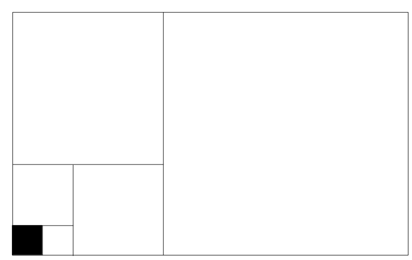
which is a train number. Try the same thing with another triple. Formulate the identity in general. Now find a train-theoretic argument that it always holds. You are welcome to work with a particular example such as the one above.

[Note to the teacher: this identity can be established by induction, but it is not a simple argument. Try it.]

6. Note that: $3\times8 + 5\times13 = 89$ which is a train number.
(a) Formulate the identity in train notation and then formulate it in general.
(b) Now find a train-theoretic argument that it always holds. You are welcome to give the argument using the particular example above.

7. Verify that

$$8^2 + 5^2 + 3^2 + 2^2 + 1^2 + 1 = 8 \times 13$$



(a) Show that a geometric proof of this identity can be obtained from the picture at the right.

(b) Find a "trains" proof of the formula. [This is a lovely problem. Use two trains and park them side-by-side.]