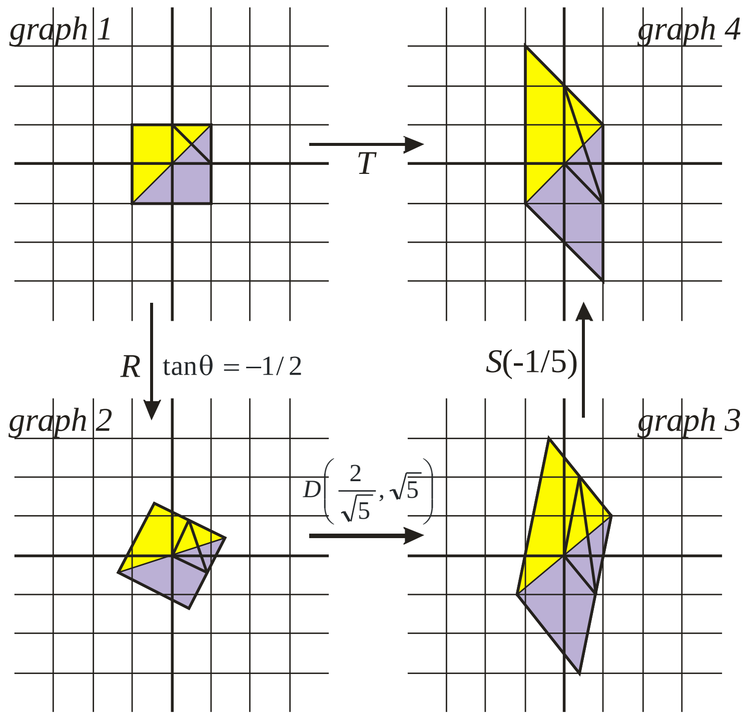
Introduction to Transformations

This unit works with linear transformations and our objective is to "factor" these as a "product" of "basic" transformations. The language suggests a parallel with prime factorization and indeed when we come to the algebraic version we will see a composition of basic transformations turn into a product of elemental matrices. That's a beautiful piece of structure and that's the main reason this is one of the projects we are developing.

Here's an example:



The transformation *T* at the top is given to the student who must construct it out of the basic transformations and typically the recipe is given as well: use a rotation followed by a dilation followed by a shear. It looks challenging, but there are ways of analysis the student learns, and there's some nice geometry to go along with it.

In fact there's some nice algebra too and the students will solve this problem in two different ways, one geometric using the diagram, and the other algebraic after the diagram has been reformulated in the language of matrices. It's a great experience to get the same set of answers through two quite different but parallel paths.

Linear transformations really belong in university math and when I work with these in my first-year course, I give the students proper definitions and we need to prove a number of fundamental theorems in order to fully understand what's happening. My objective here is rather different. The transformations are sophisticated objects that the students will work with and play with as way of developing their mathematical thinking and their imaginative capacities. This is analogous to an English student working with a sophisticated novel or play, or a music students with a concerto, or indeed any student in the creative arts. And mathematics of course is both a science and a creative art. Here I am doing exactly what Dan Kennedy advocated in his *Climbing around the tree of mathematics*—parachuting my students up onto the leafy ends of the branches where all the exciting activity goes on.

The trick is that for my students to work with these things, they have to have enough of a handle on them. Constructing a workbook that delivers that is a bit of a challenge with both a technical and an intuitive dimension—enough drill, but not more than is needed, and a collection of examples that is both accessible (low floor) and sophisticated (high ceiling). This is a balancing act and when I have erred it has always been on the side of too many details and too much drill. Let them play, and talk with them when they are stuck.

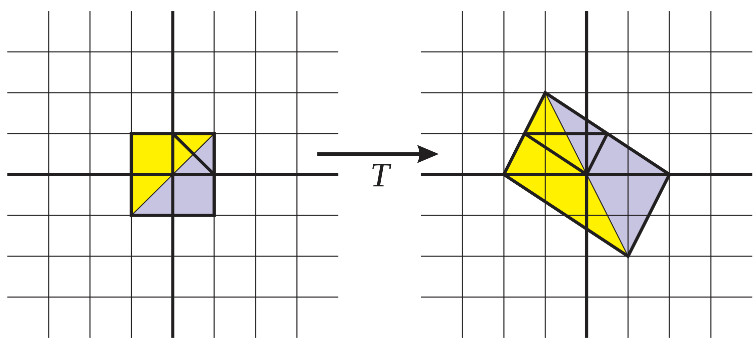
"Talk with them when they are stuck." That suggests an important role for tutors and indeed I usually make good use of these, either my own graduate students, or students from a senior grade, or even (and especially) students in the class working with peers.

A good example of the light technical touch is found in my introduction at the beginning of the workbook in which I give them enough of a definition of linear transformation to allow them to proceed to the examples. I used to say quite a bit more, but I am now convinced that it is more important to get them started early on the factorization problems.

It might seem to some that this philosophy violates the central principle of John Mighton's JUMP Math program, but I think not. Progress in mathematics is a balancing act between forging ahead to get a better grasp of the landscape and then retreating to bear down and consolidate our understanding of the tools we have found that we will need. This is true not only in mathematics but in all branches of learning and no one has laid this out better than Whitehead in his 1929 book *Aims of Education*.

This manual is designed to be used along with the student workbook. It provides a commentary on the Examples and answers to the problems. One last point—some students will catch on quickly and will want to forge ahead. Sometimes I get them to work with others; other times I given them some new problems. Here's an early (in the workbook) example of such an "extension." Use your intuitive understanding of the linearity of T to find the T-images of the following points P. The answers are given in the second column.

P	T(P)
(0, 1)	(–3/2, 1)
(–1, 0)	(–1/2, –1)
(1/2, 1/2)	(–1/2, 1)
(–1/2, 1)	(–3/4, 1/2)
(0, 2)	(–3, 1)
(2, 1)	(–1/2, 3)



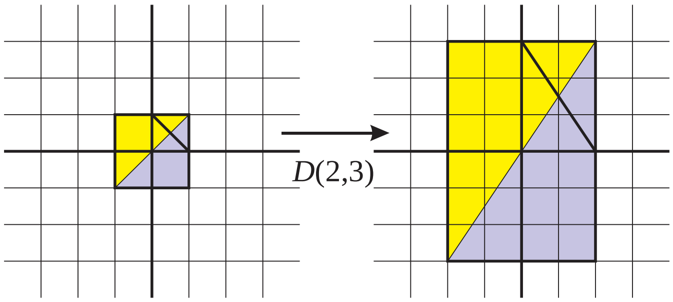
Comment. In fact in grade 10 these students have hardly met the concept of a function. In fact the term "function" is not used in the grade 9 or 10 curriculum; rather it is the opening concept of the grade 11 course. What we do have in grades 9 and 10 is the concept of *relation*: An identified relationship between variables that may be expressed as a table of values, a graph, or an equation.

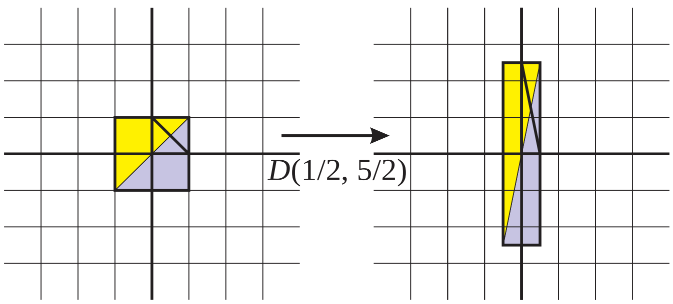
So they do have tables of values and graphs. My own feeling is that diagrams such as the one above might be a better visual way to be introduced to a function than the standard one-variable function graph. The graph, say of a quadratic polynomial seems less friendly to me than visual representation of the function T above. Evidence for that might come from the difficulty students have working with distance-time graphs. So maybe this unit provides the right occasion to encounter functions!

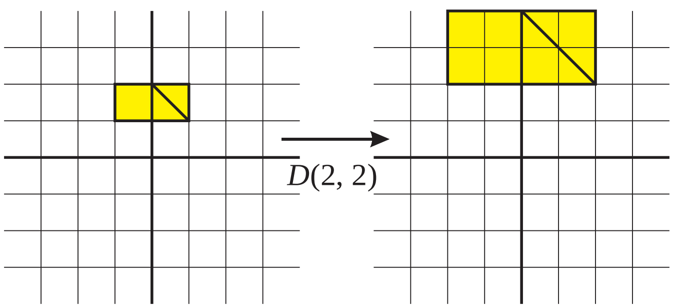
The Basic Transformations

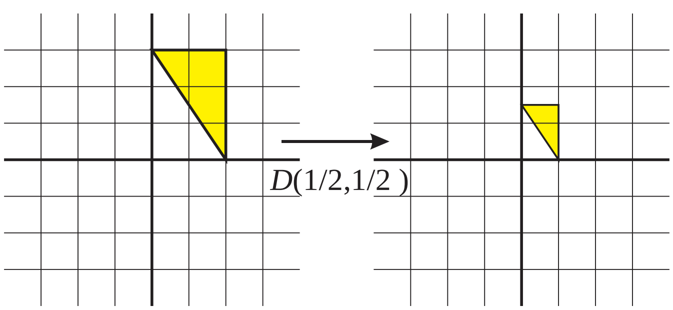
Dilations

In each case, the student is asked to provide the diagram in the right-hand grid.

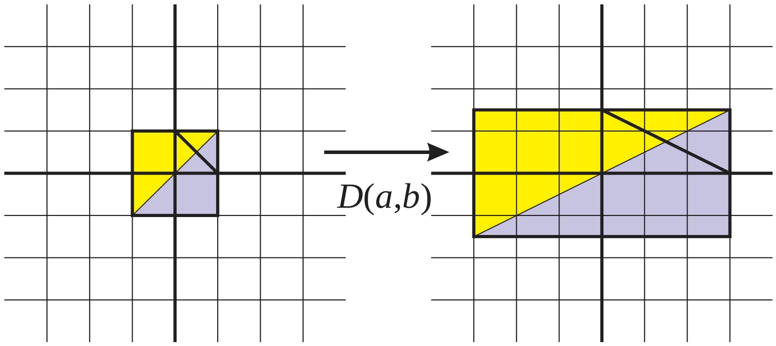




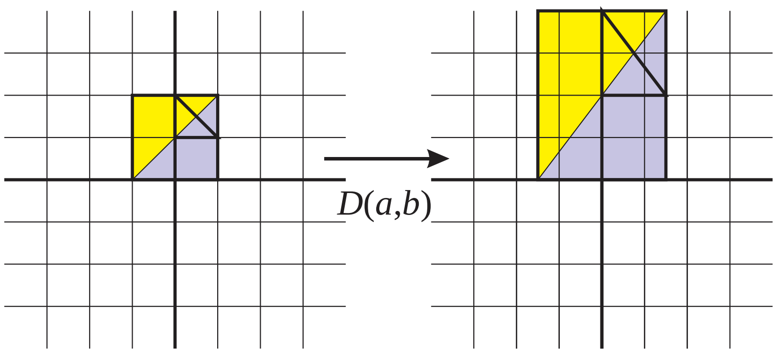




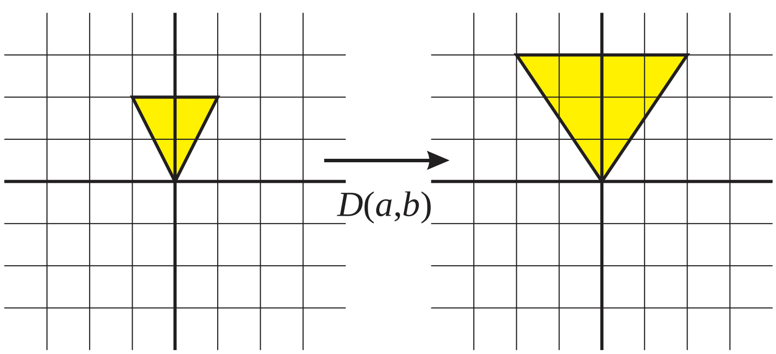
In each case the student is asked to calculate *a* and *b*



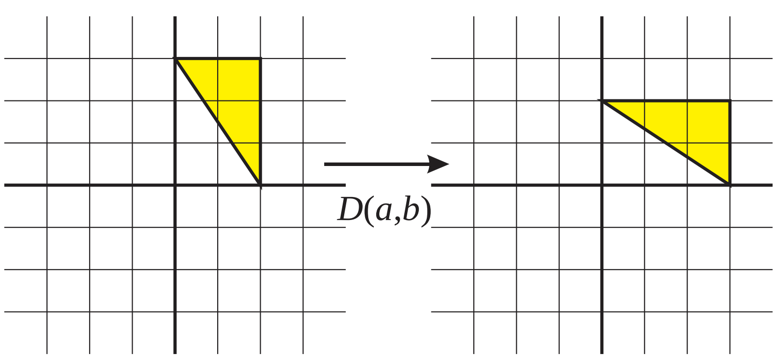
$a = 3$
$b = 3/2$



$a = 3/2$
$b = 4$



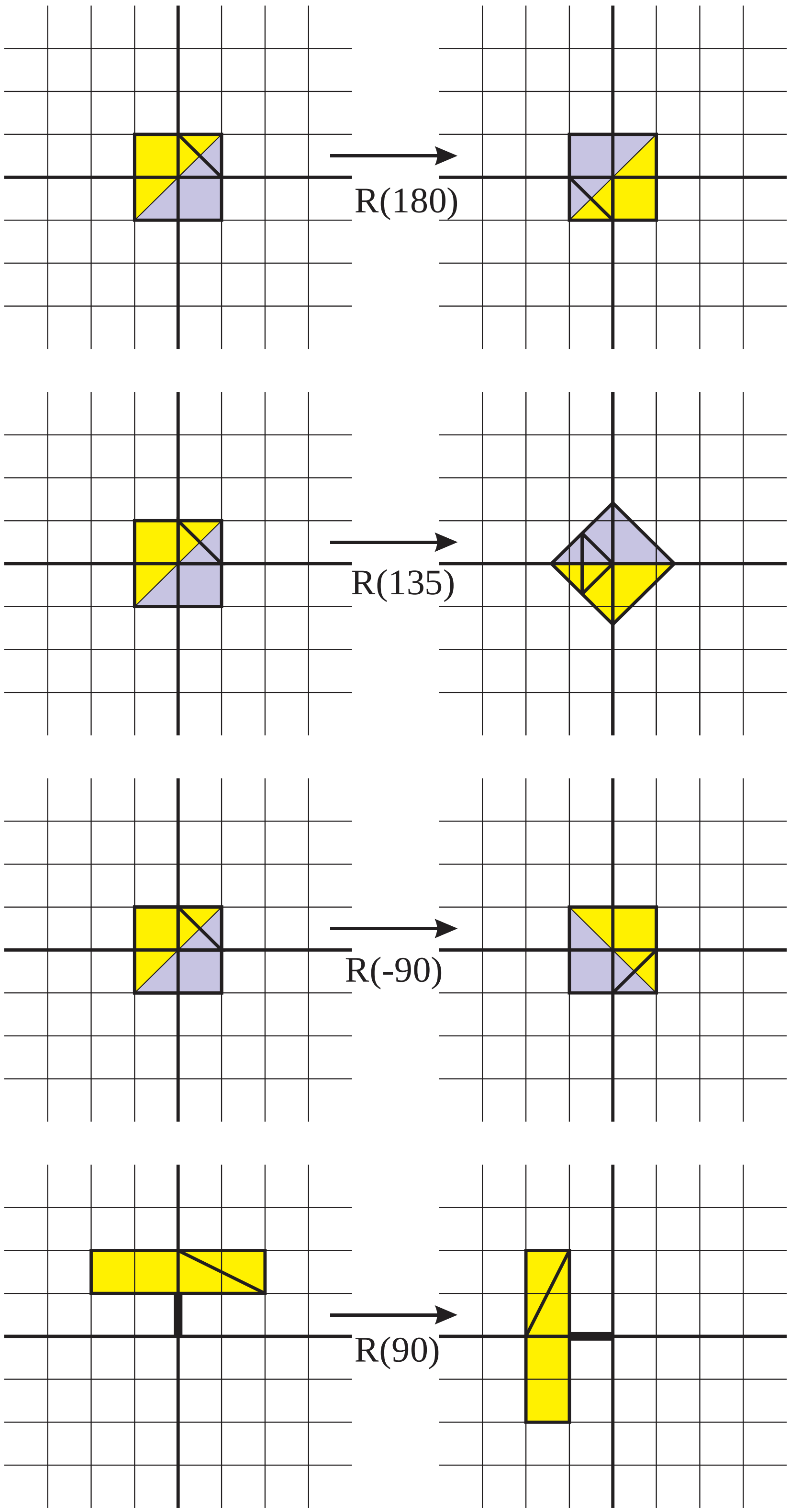
$a = 2$
$b = 3/2$



$a = 3/2$
$b = 2/3$

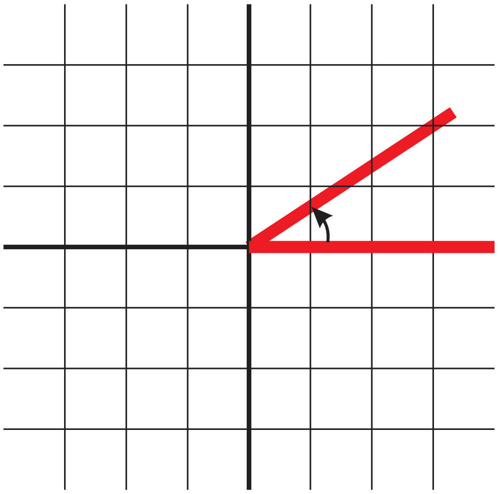
Rotations

In each case, the student is asked to provide the diagram in the right-hand grid.

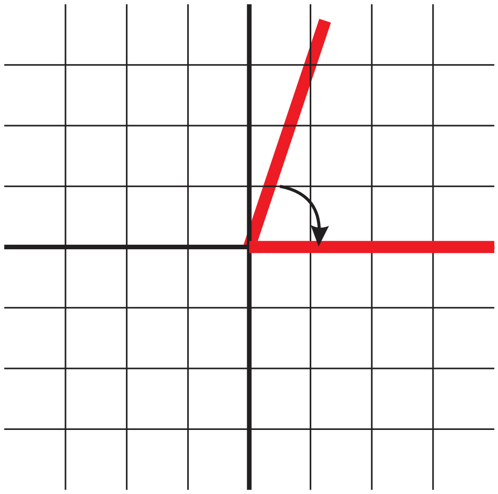


Notice that I put a "handle" on it. Makes it easier to track.

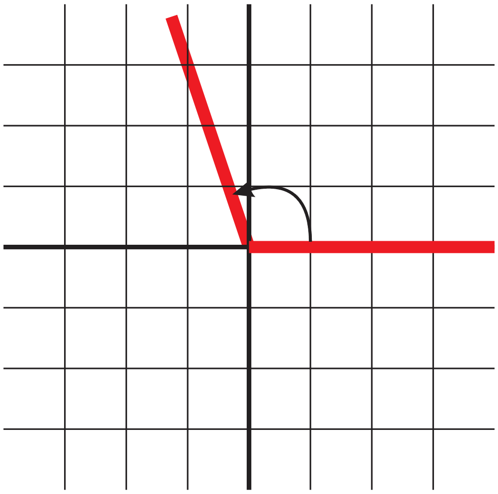
In each case the student is asked to calculate the rotation angle θ.
[When a line seems to go through a grid point assume that's exactly the case.]



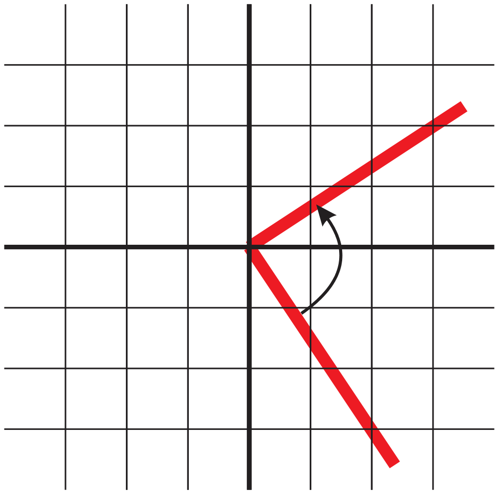
$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.69°$$



$$\theta = -\tan^{-1}(3) \approx 71.57°$$



$$\theta = 90° + \tan^{-1}\left(\frac{1}{3}\right)$$
$$\approx 90 + 18.43° = 108.43°$$



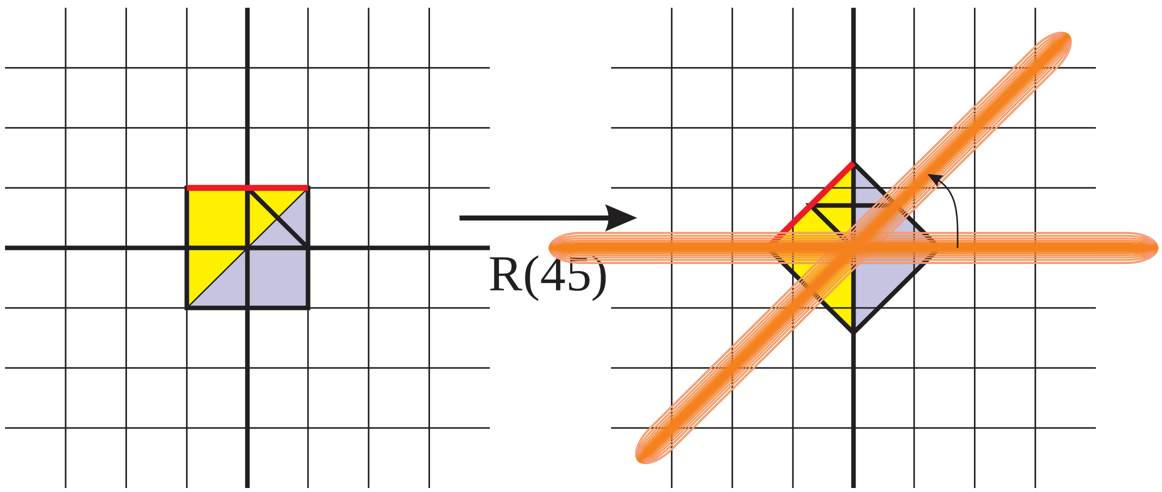
$$\theta = \tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{3}{2}\right)$$
$$\approx 33.69 + 56.31 = 90°$$

That's interesting: can you see a reason why this should be a right angle?

Hands-on!

I have always thought that the shear is the most complex of the basic transformations but we have been told (and have noticed) that some students have more trouble with the rotation. I do know that there is great variation among students in their capacity for spatial reasoning. Here's a "hands-on" exercise that seems to help in understanding that the rotation in the example above is 45º. Take a coffee stir stick and lay it along the *x*-axis in the target graph (on the right). Thus it is parallel to the top of the box in the starting graph. Now rotate it about the origin until it is parallel to the footprint of the top of the box in graph 4. What angle has it turned through?

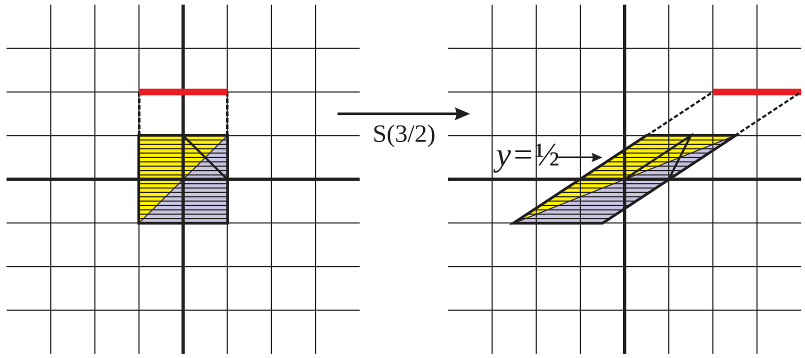
Even better—use two sticks and rotate one of them. It can help a lot to have the starting image and the rotated image on the same graph.



The shear

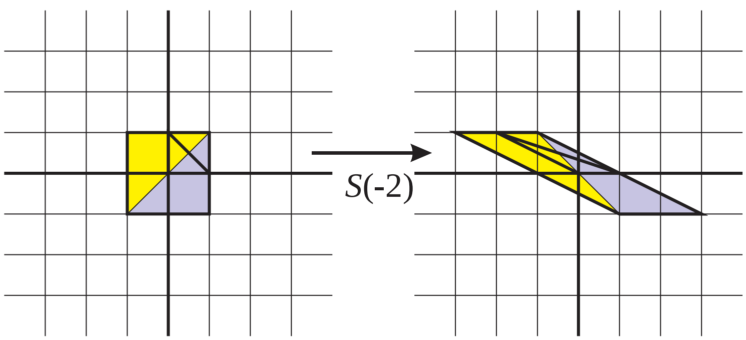
The best way to understand the shear is to imagine the square made up of horizontal lines. When we apply the shear $S(h)$, each such line is moved sideways (horizontally) but lines at different heights are moved different amounts. In fact *the amount the line is moved is proportional to its height above the x-axis (its y-coordinate)* and h is the constant of proportionality. In the example below I have taken $h = 3/2$.

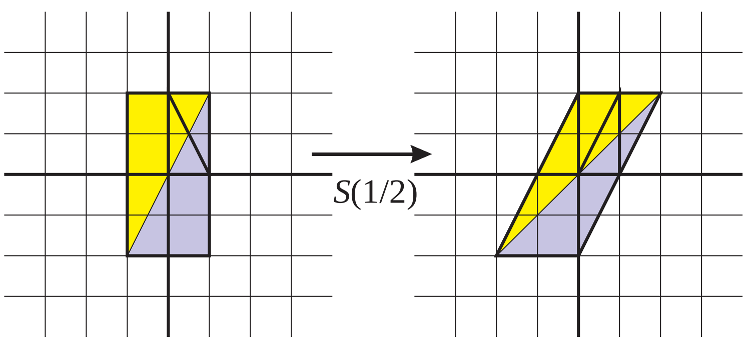
Points with $y = 1$ are moved a distance h.
Points with $y = ½$ are moved a distance $h/2$.
Points with $y = 2$ are moved a distance $2h$.
Points with $y = –1$ are moved a distance $–h$.
Points with $y = y$ are moved a distance hy.

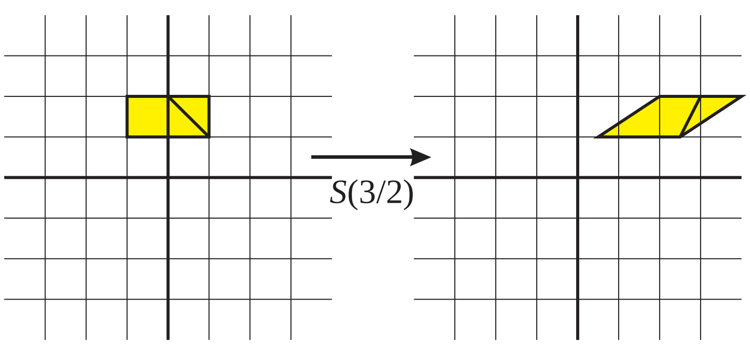


For example, under $S(3/2)$ a line at height ½ will move a distance (3/2)(½) = ¾ and a line at height 2 will move a distance (3/2)(2) = 3.

In each case, the student is asked to provide the diagram in the right-hand grid.

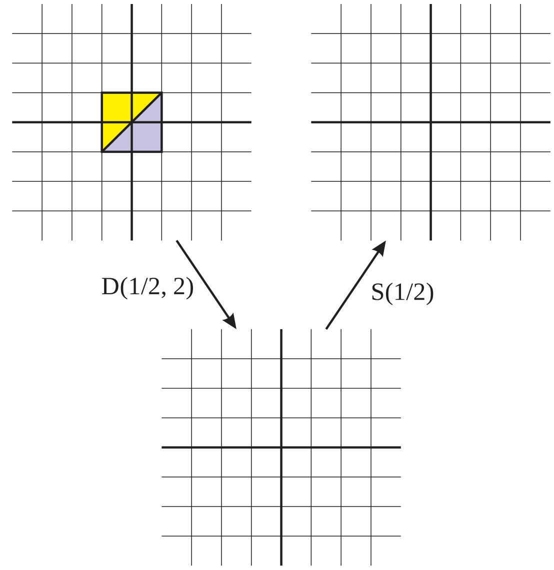


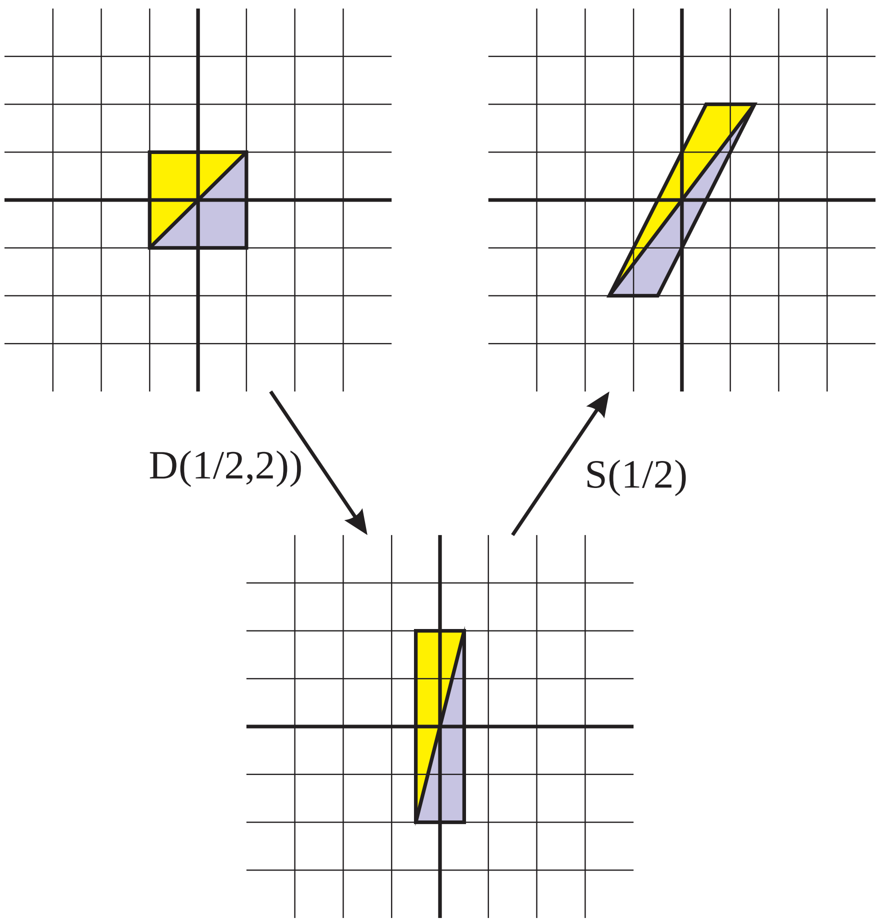




A two-step transformation.

The student is asked to provide the missing diagrams





Can you get to the same endpoint with a shear first followed by a dilation?