Coffee&Milk

You buy a large coffee (300 ml) at 80° and four milks (15 ml each) at 4°, and get into your car which stays at a constant temperature of 16°. It will be a 10 minute drive before you can drink the coffee, and you want it to be as hot as possible at that time. Do you put the milk in at the beginning or after the 10 minutes when you're ready to drink it? Do you mix first and then drive or drive first and then mix? Does your intuition give you an answer?

Mix last. Let's start by keeping them separate till the end. Then what will happen to each during this 10-minute period? Well the coffee will cool down. Why?––because it's hotter than the inside of the car. And the milk will warm up because it's cooler than the car. Okay. It seems we'll need to know something about just how fast those two processes occur. Can we get a bit more information here?

Sure. Let me tell you precisely what happens to the temperature of each if they are kept separate. After 10 minutes in the car, the coffee drops from 80° to 76° and the milk rises from 4° to 7°.

Okay. Then when we mix them, we ought to get some sort of average of 76 and 7. But what sort of average?––a straight average? Well, no. There's much more coffee than milk so it should be weighted more. How much more weight? Maybe the weights should be proportional to the amounts. In fact that's exactly right: when you mix two similar liquids at different temperatures, the final temperature is the weighted average of the two where the weights to use are the relative amounts.

Here's the principle. If liquid 1 has temperature T_1 and volume V_1 and liquid 2 has temperature T_2 and volume V_2, then when they are mixed the resulting temperature will be the weighted average:

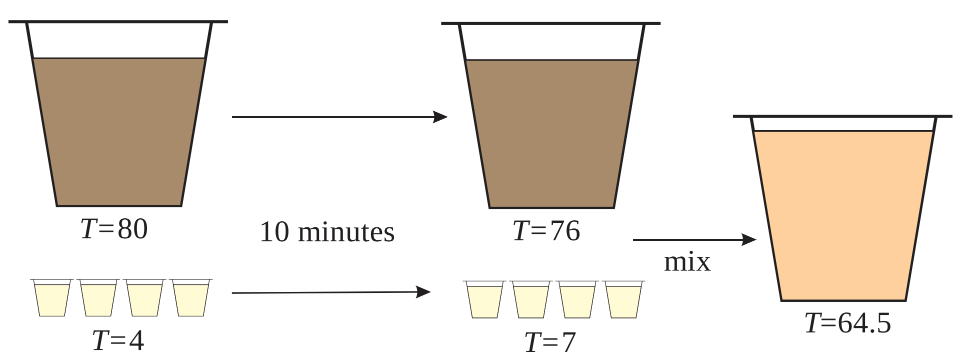
$$T = \frac{V_1 T_1 + V_2 T_2}{V_1 + V_2}$$

An expression like this is called a *weighted average*. The V_i are the weights and when we divide by the sum of the weights, we get an average of the T-values.

Note that this simple volume weighting only holds when the liquids are the same or at least have the same "specific heat." For our purposes, coffee and milk are both mainly water, so we treat them as being the same substance.

Thus, given 300 ml of coffee at 76° and 60 ml of milk at 7°, the final temperature for the "mix at the end" strategy is the weighted average:

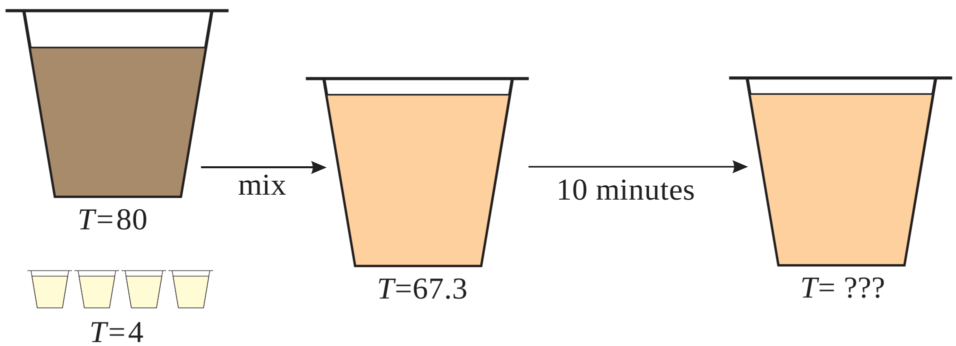
$$T = \frac{300(76) + 60(7)}{360} = \frac{23220}{360} = 64.5$$



Mix first. If we mix at the beginning, we'll right away get the same type of weighted average but now using the starting temperatures 80° and 4°:

$$T = \frac{300(80) + 60(4)}{360} = \frac{24240}{360} = 67.3$$

So this is the temperature of the mixture at the beginning. The question is, what happens to it over the 10 minutes it sits in the car?



Hmm. The only temperature change information we have so far pertains to the coffee and milk separately. Is that of any help to us once we mix them?

To penetrate this, we have to understand the law of temperature change. If you put a hot object (temperature T) into a cool environment (held at a constant temperature E), the object will lose heat energy to the environment and its temperature will go down. And as you might expect, the *rate* of this temperature decrease will depend on the temperature *difference*

$$D = T - E$$

between the object and the environment. In fact it will be *proportional* to this difference.

A consequence of this is that the temperature difference D decreases at a constant *percentage* rate. And that makes D an exponentially decaying function of time

$$D(t) = D_0 r^t$$

where D_0 is the value of D at the beginning (t=0) and r is the factor by which the difference is multiplied in each minute.

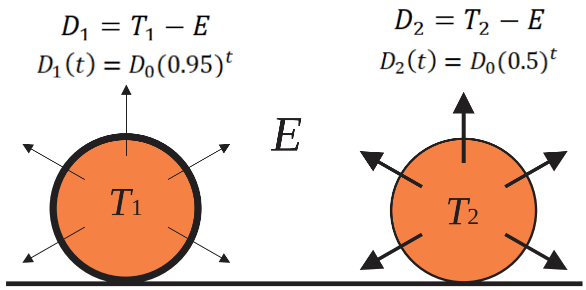
We call r the one-minute multiplier, but keep in mind that it is the *difference D* that gets multiplied.

Law of temperature change.

Suppose a hot object sits in a cool environment held at constant temperature E. Then the object will cool down at a rate which is proportional to the difference in temperature between object and environment.

This is a fundamental principle known as Newton's Law of Cooling. This law also holds for cool objects in a warmer environment.

This principle is illustrated in the diagram at the right. We put two large marbles on the table in a room held at a constant temperature *E.* Both start at the same temperature, D_0 degrees above the temperature of the room. As time goes on the temperature difference D_i between each marble and the room decays exponentially. Now marble 2 is insulated and marble 1 is not. Thus marble 2 loses its heat slowly, in fact the difference D_2 loses only 5% of its value every minute. On the other hand, marble 1 loses heat quickly and the temperature difference D_1 is cut in half every minute.



Now back to our cup of coffee. We have put the milk in the coffee and now we need to drive for 10 minutes. What will happen to the temperature?

What we have learned is that the temperature difference between the coffee and the car will be multiplied by some factor *r* over each of those minutes, and so over 10 minutes, the multiplier will be r^{10}. But how do we find *r*?

Well *r* depends on the effectiveness of the insulation, and the data we were originally given provide a couple of *r*-values, one for the coffee cup and one for the plastic milkers. Let's work them out.

Actually we'll only need the 10-minute multipliers and those can be worked out from the tables at the right.

Coffee cup		
t	*T*	*D*=*T*−16
0	80	64
10	76	60

Milker		
t	*T*	*D*=*T*−16
0	4	−12
10	7	−9

The coffee-cup: In 10 minutes, *D* is multiplied by

60/64 = 15/16.

The milker: In 10 minutes, *D* is multiplied by

−9/−12 = 3/4.

In 10 minutes, the coffee-cup difference loses 1/16 whereas the milker difference loses 1/4. That makes sense––the coffee cup is much better insulated than the small plastic milkers.

Which *r* value do we use for the 10-minute drive? Well since the mixture sits in the coffee cup, we should use the coffee-cup value. The 10-minute multiplier is

$$r^{10} = 15/16$$

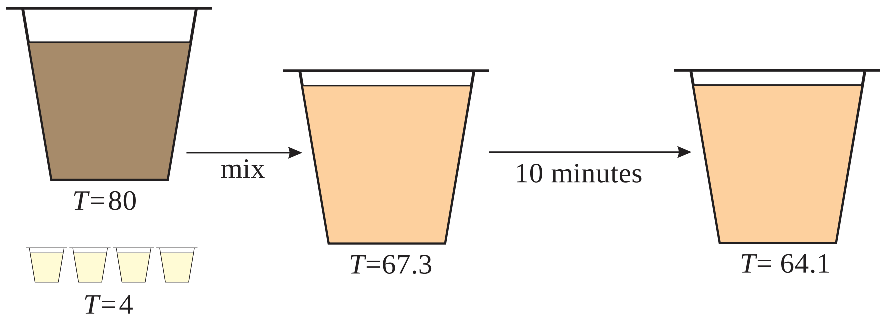
Let's do the calculation.

Right after mixing: $T_0 = 67.3$

$D_0 = 67.3 - 16 = 51.3$

After 10 minutes: $D = 51.3\frac{15}{16} = 48.1$

$T = 48.1 + 16 = 64.1$



What's the final verdict? The final coffee temperatures are:

Mix last: $T = 64.5$

Mix first: $T = 64.1$

Better to keep them separate in the car and mix just before you drink. That strategy gains you a whole 4/10th of a degree!

Was it worth all that work?––for a measily difference of less than half a degree? Sure it was. It gave us a great math problem.

It worth trying to understand what accounts for the difference. The coffee is hotter than the car and the milk is colder than the car, so over the 10-minute period, the coffee will cool down and the milk will warm up. Since we want the final mixed temperature to be as high as possible, we want the coffee to cool down as little as possible and the milk to warm up as much as possible. So we want the coffee to be well-insulated and the milk to be poorly insulated.

Now the coffee is always in the coffee cup so that determines its insulation no matter what we do. But the milk passes the 10-minute period in two different ways: if we mix last it's in the milker and if we mix first it's in the coffee cup. If we want it to be as poorly insulated as possible, it needs it to be in the milkers. So we should mix last!

There's the answer. To maximize the final temperature of the mixture, the milk should sit in its plastic containers while you are driving.

Is that what your intuition told you?

I've worked many problems with teachers over the years. And when I meet them at a conference after a long time, this is often the problem they remember. Why? Because they've "been there, done that."

Problems

1. I have a portable heating coil which I can plug in (say in a motel room) and put into a large mug of water and heat it up to make tea. If I pop a super tea cozy over the mug while I'm doing this, then there is no heat loss to the room, and the water temperature increases at a constant rate, going from 20° to 100° in 5 minutes.
(a) Draw a rough graph of the water temperature against time.

(b) Now suppose that I've got the water to 100°, but instead of making the tea, I get waylaid by a desire to perform a couple of experiments. I begin by simply leaving the mug standing in the room without the cozy. Draw what you think is a reasonable graph of the temperature of the water over a 4-minute period. Here are a couple of points you can mark on your graph––during the first two minutes, the temperature drops from 100° to 70°, and during the second two minutes it drops to 50°.

(c) Now, with the water temperature at 50°, I put the heating coil back in the mug and attempt to heat it back up to 100° *but without the cozy*. Do I succeed? No! I find that after a long time the temperature of the water stabilizes at a level somewhat short of boiling. Show that an analysis of your two graphs will allow you to predict the temperature at which my mug of water will stabilize.