Optimal driving speed

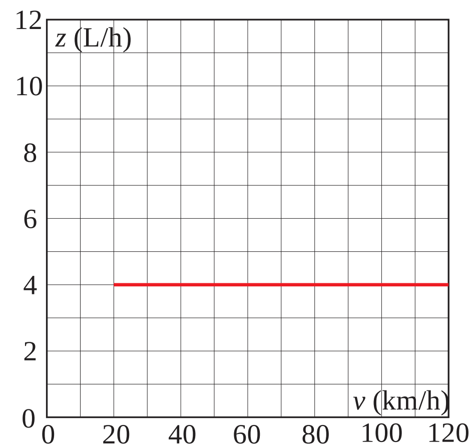
Problem 1. A friend has offered to drive your car to Vancouver so it will be waiting for you when you get off the plane. She is in no hurry, so is happy to drive at any speed you want, provided that you pay for the gas. What you want to know is how fast to ask her to drive to minimize your costs.

To answer this, you'll need to know how the gas consumption of the car depends on the speed v (km/h) at which the car is driven. Now there are two ways to measure the rate of gas consumption, litres per hour and litres per kilometer and it's important to notice which you are talking about. We will work with both of these but our fundamental variable z will measure litres per hour. Thus, if you drive at constant speed for exactly one hour, then v will the distance traveled and z will be the amount of gas used.

In the first section we work with the graph of z against v and find a graphical analysis of the problem. In the second section we build a formula for z in terms of v and use calculus.

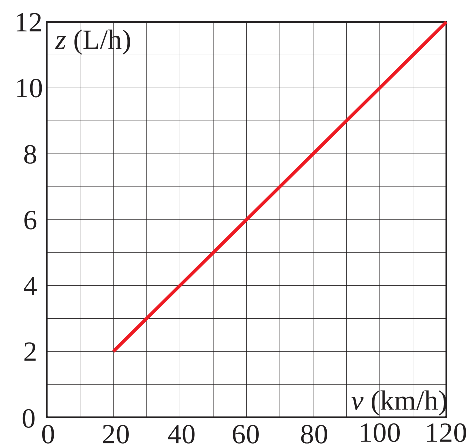
1. Working with the graph

First we need to decide what the graph looks like and to get the discussion going I put up the simplest possible candidate––a constant function. What about that?



I get universal protest.
If you drive faster you use more gas.
What does that mean? Be more precise.
You burn gas at a higher rate. The graph has to increase.
That is correct. Everyone okay with that?

I get a question. *Why am I starting the graph at v = 20?*
That's a good question and it gets them talking about gears. At different speeds you use different gears and that affects gas consumption. I have decided to ignore that as most of the driving is on the highway so I assume that we use only high gear. And you can't go real slow in a high gear.

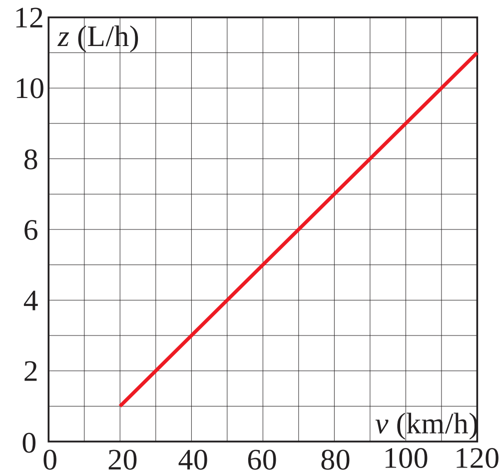


Okay. I give them an increasing graph. They stare at it for a while. They are clearly uncomfortable.

There is a good discussion and the issue is identified. If you drive at 20 clicks for an hour you use 2 litres; if you drive at 100 clicks you use 10 litres. In the first case you go 20 km on 2 litres of gas and in the second case you also go 20 km on 2 litres of gas. But is it not more expensive to drive faster?

Very good. That is indeed the case and is an effect we will look at more closely later.

I respond to these points with the offering on the right. It's still a straight line but I wonder if it will satisfy them. Does this graph make it more expensive to drive faster?



One group of students makes the endpoint calculations:

$$v = 20, z = 1 \qquad \frac{\text{L}}{\text{km}} = \frac{1}{20} = 0.05$$

$$v = 120, z = 11 \qquad \frac{\text{L}}{\text{km}} = \frac{11}{120} = 0.09$$

You certainly use more gas per km at v = 120 than at v = 20, almost twice as much.

Yup. Does this hold for all speeds? How would you tell? The class has to think about this and I let them talk. They see that what we have to calculate is the quotient z/v:

$$\frac{z}{v} = \frac{\text{L/h}}{\text{km/h}} = \frac{\text{L}}{\text{km}}$$

How do we do that? Well we can actually find the equation of the line. It has slope 10/100 = 1/10 and it passes through the point (20, 1):

$$z - 1 = \frac{1}{10}(v - 20)$$

$$z = \frac{v}{10} - 1$$

I have used the point-slope form of the equation of a line but my students always use the slope-y-intercept form:

$$y = mx + b.$$

First they calculate m = 1/10:

$$z = \frac{v}{10} + b$$

and then they plug in the point (2, 1) to get b.

Then litres per km is:

$$\frac{z}{v} = \frac{v/10 - 1}{\text{v}} = \frac{1}{10} - \frac{1}{v}$$

This is indeed higher when v is higher. The faster you drive the more gas you use for any fixed trip.

Okay. Is this a reasonable graph then? I take a vote.

The nays win. Most students do not like this graph. They tell me it should not be a straight line, and a number of reasons are given.

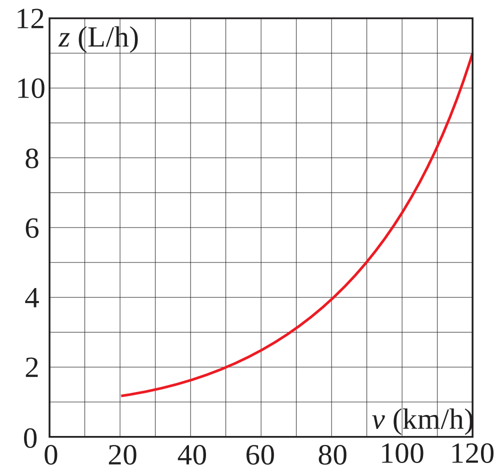
First at high speeds gas consumption should go up quite fast, but for this graph, v has very little effect on z/v as $1/v$ is very small. Another student says that the graph should be steeper for larger v––and that's really the same thing.

Another student observes that gas consumption (L/km) will actually go up if you drive too slow, that engines aren't built to run all the time at 20 km/h. An interesting observation.

Ah—very good. That last remark raises the question of engine efficiency. That's an interesting factor that we'll talk about later.

Okay. They are right. The graph should not be straight. But should it be concave-up (smile) or concave-down (frown)? Again I take a vote and the majority goes for the smile. That makes sense because we want a steeper slope at higher *v*.

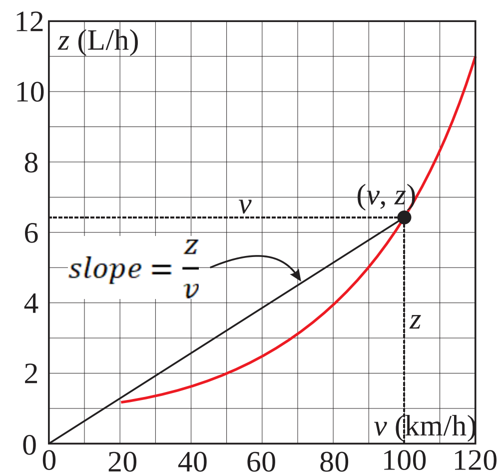
Indeed the graph is concave-up. That gives us an extra penalty for going too fast as well as an extra penalty for going too slow. We will be looking at each of these effects in more detail when we come to the question of finding an algebraic formula for *z* in terms of *v*.



It's time to display the graph that we will be using. It's constructed from automobile specs (of some years ago). I distribute a copy of the graph and ask the students to find the speed that minimizes the gas consumption for the Vancouver trip.

What they need to minimize is the number of litres used per km and we have seen that that's the quotient:

$$\frac{z}{v} = \frac{\text{L/h}}{\text{km/h}} = \frac{\text{L}}{\text{km}}$$



But how do we calculate this without an equation for the graph? How do we compare L/km at different points on the graph?

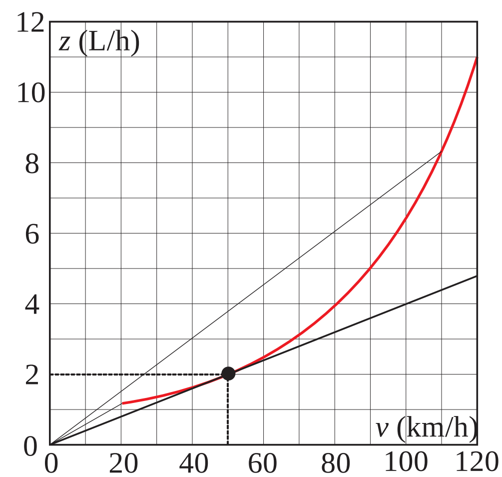
Well there's a nice simple geometric interpretation of z/v, but it takes them a few moments to see it.

It's the slope of the line drawn from the origin to the point.

Yes it is. For any point on the graph, z/v is the slope of the line drawn from the origin to the point:

$$\frac{z}{v} = \text{slope of secant from } (0,0) \text{ to } (v,z)$$

Nice. Now for which point on the graph does this secant have the smallest slope?

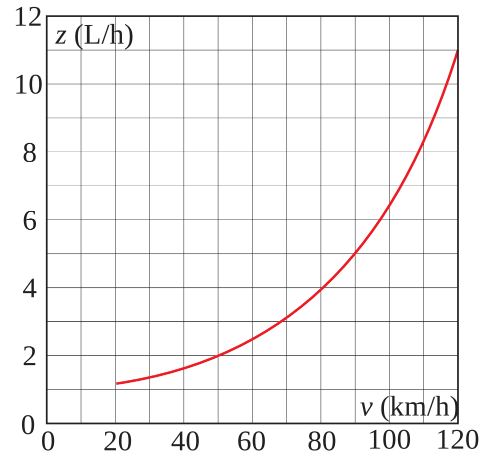


For this graph, this clearly occurs at the tangent to the curve drawn from the origin. That gives us an approximate optimal value of *v* = 50 km/h using 2 litres of gas per hour.

Your friend should cross the country at 50 km/hr. That'll give her a real sense of the size of the great prairie country.

2. Putting a value on time

I ask the students whether *they* would drive across Canada at 50 clicks.
No way
So why not?
Takes too long. Better things to do.
Yes time is short. So let's see what happens if we put a value on time spent.



Problem 2. Suppose now that, in addition to paying for the gas, you have also agreed to pay your friend a "wage" of $6 per hour of driving. If the price of gas is $1 per litre, what driving speed will minimize your total cost?

Now your total cost has two components, both of which are given in terms of cost per hour:

$$\text{gas} = (1.00)z = z \qquad \left[\frac{\$}{\text{h}} = \frac{\$}{\text{L}} \cdot \frac{\text{L}}{\text{h}}\right]$$

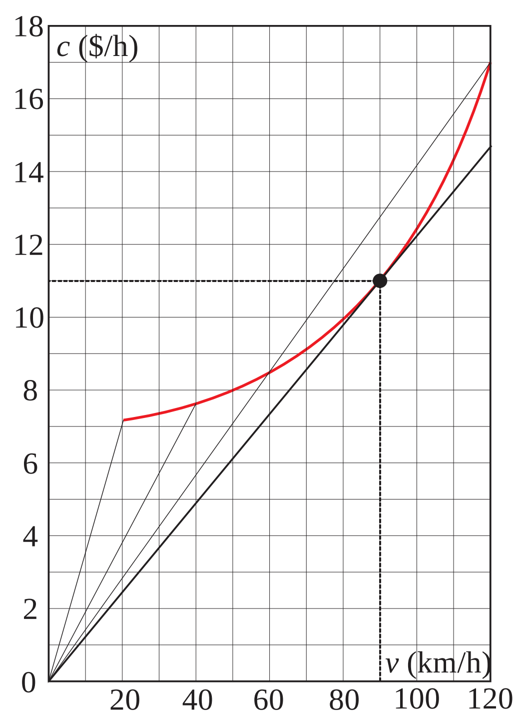
$$\text{wages} = 6 \qquad \left[\frac{\$}{\text{h}}\right]$$

The total cost per hour *c* is the sum of these, and the total cost per km is obtained by dividing cost/hour by the speed *v*:

$$\frac{\text{cost}}{\text{km}} = \frac{c}{v} = \frac{z+6}{v} \qquad \left[\frac{\$}{\text{km}} = \frac{\$/\text{h}}{\text{km/h}}\right]$$

This is minimized when $\frac{z+6}{v}$ is a minimum.
Can we interpret that graphically?

Well if we had a graph of *c* against *v* we could do the same thing with it that we did with the *z*-graph––draw the tangent to the graph from the origin. Now the graph of *c* = *z*+6 is obtained simply by lifting the *z*-graph 6 units. The result is diagrammed at the right.



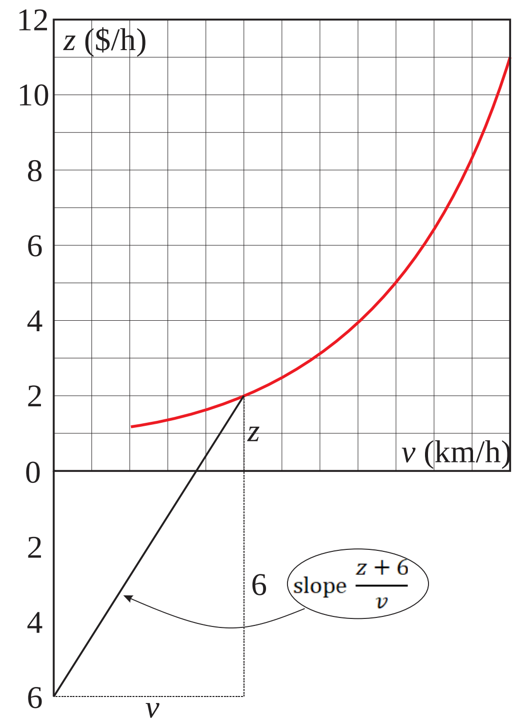
The minimum slope is obtained at *v* = 90 for a cost of $11/h of which $5/h belongs to the gas. Notice that the vertical axis is now *c*, the hourly cost.

Let's go back to the students sitting in the classroom. They don't have the *c*-graph at their desks––they only have the *z*-graph. How are they to get the solution?

They can of course construct the c-graph simply by extending the vertical axis down 6 units and renumbering the vertical axis. But most often they simply keep thinking of it as the z-graph.

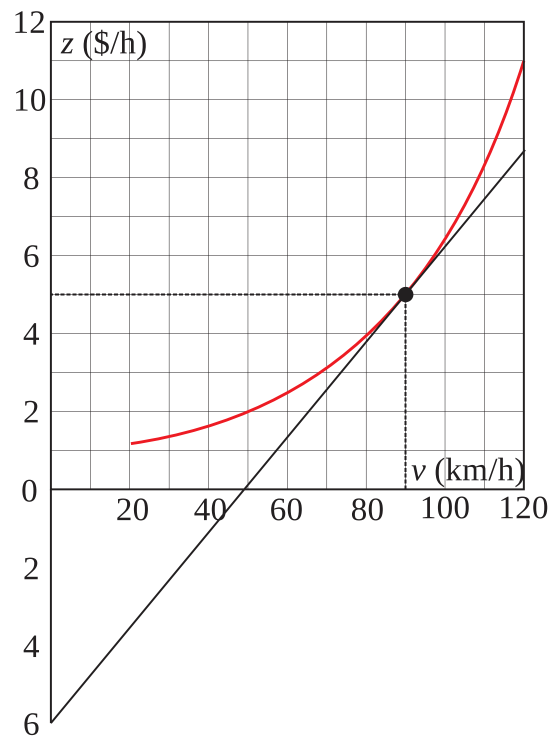
To work with this, they have to convince themselves that if they draw the secants to the graph from a point 6 units below the origin, the slope of the secant will be $(z+6)/v$. The diagram at the right emphasizes this.

Give that it is clear that their job is to find the (v, z) point which makes this slope a minimum.



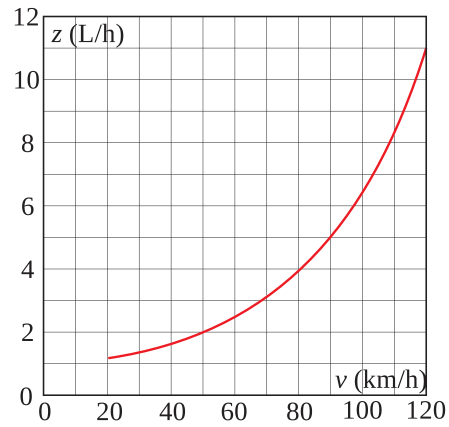
Note: The gas cost I used of \$1/L is a bit special in that it allows us to work directly with the z-graph as it represents both litres and dollars. Thus, the z-graph itself gives us gas dollars and the extension of the axis below gives us wage dollars. If the unit gas cost was \$2/L we would simply have doubled the values on the z-axis (but not of course on the lower extension).

And then it is clear that this achieved with the tangent. That gives us the same solution we had with the c-graph.



3. A model for gas consumption

Our purpose here is to construct a simple algebraic model for the gas consumption graph of the driving speed problem. Using energetic considerations, we try to work out how the gas consumption z (L/h) of the car (in high gear) should depend upon its speed v (km/h). Of course we'll assume that we have a level road, with no wind, etc.



Well why does it take energy to move a car along a level road? One important factor is certainly air resistance. How can we quantify that?

I find it is well worth spending a half hour of class time talking about these ideas. In fact most students have a only a weak grasp of the fundamental ideas concerning work and energy, and will find the prospect of trying to put together a model for gas consumption quite intimidating. But I am often impressed by how much the students manage to do with minimal prompting from me. And at the end, they are often also impressed by what we have all accomplished.

Air resistance. Well, consider a cylinder of air of length 1 km and cross section the same as that of the car. Your car, in traveling 1 km, will bang into all the air molecules in that cylinder, and in doing so will give them all some energy of motion (kinetic energy) and that's energy that the fuel must provide. Now the question is how that amount of energy should depend on the speed v of the car?

Well, the higher v is, the higher will be the speed that you impart to those molecules. Of course the speed each molecule gets will depend on whether it's hit head on, or given a glancing blow, but, on the whole, given that it's hit in a certain way, the speed it gets should be proportional to the speed of the car. That is, if we run the car a second time through the cylinder, but with twice the speed, the speed given to each molecule should be more or less doubled.

So the speed that each molecule gets is proportional to the speed v of the car. What does that mean for the amount of *energy* that each molecule gets? Well, the kinetic energy of a moving body is proportional to the *square* of its velocity. We conclude that the energy imparted to that cylinder of molecules is proportional to the square of the speed v of the car.

Kinetic energy is proportional to the square of velocity

Perhaps some of you have seen the formula

$$\text{KE} = \frac{1}{2}mv^2$$

which says that the constant of proportionality is half the mass.

Now here's a place to be careful about units. What this says is that energy *per kilometer* (that's the length of the cylinder of air) is proportional to v^2. Of course, what we want is energy *per hour*. Now to change from energy/km to energy /h we multiply by km/h, i.e. by the speed of the car. When we do that, our v^2 becomes v^3.

Air resistance: $$\frac{\text{energy}}{\text{h}} = \frac{\text{energy}}{\text{km}}\frac{\text{km}}{\text{h}} \propto v^2 \cdot v = v^3$$

So to combat air resistance, the rate (per second) at which you burn gas is proportional to the *cube* of the car's speed.

For example, compare making a trip at 60 km/h and 120 km/h. At the higher speed you burn gas at 8 times the rate of the lower speed. But at 120 you drive for only half the time so the consumption for the whole trip is only 4 times as much.

That's air resistance. What else is there?

That's taking account only of air resistance. But there's more.

Rolling resistance. Imagine pushing a car at a fixed speed along the road with no air resistance (maybe there's a gentle breeze with the same speed as the car or maybe you're on the moon). Do you still have to exert a force? Well, yes you do; the force that you have to work against is called rolling resistance, and comes mainly from the resistance of the tires to rotate. For example, this resistance can be reduced by increasing the pressure in the tires (try it with your bicycle!), and will be increased if you fill the car with people. The question is how this rolling resistance depends on the speed of the car.

Well, that's not so easy to guess at. Imagine pulling a toboggan laden with little kids. The force to propel it (at constant speed) is provided by the tension in the rope. The question is, how does this tension change with the speed of the toboggan. Is the force you have to exert on the rope greater at a greater speed? The answer turns out to be (approximately) NO. The force you exert is more or less independent of the speed. Now rolling resistance is just a more complicated form of force against friction, so the force of rolling resistance is also independent of speed.

What does this say about energy?––well force is energy per unit distance, so this means that that the energy you expend against friction *per unit distance* is independent of the speed. We conclude that the energy expended *per hour* is proportional to speed:

Rolling resistance: $$\frac{\text{energy}}{\text{h}} = \frac{\text{energy}}{\text{km}}\frac{\text{km}}{\text{h}} \propto 1 \cdot v = v$$

Total resistance. These two factors tell just about the whole story. Combining them, our model for energy consumption as a function of *v* is

Total energy: $$\frac{\text{energy}}{\text{h}} = av^3 + bv$$

where *a* and *b* are constants to be determined. The constant *a* is pretty well determined by the shape of the front of the car, and its cross-sectional area, and *b* turns out to be proportional to the weight of the car––back to the toboggan: the tension in the rope increases if you add a kid, and is in fact proportional to the weight of the loaded toboggan.

There's one more step. So far, we have been looking at the amount of *energy* required to move the car (per hour), but what we really want is *z* and that's the number of *litres of gas* we use (per hour). To get hold of the relationship between these two measures, we need to understand what's called the efficiency of the engine.

Force is energy per unit distance

Maybe the tension in that toboggan rope is the same whether you go fast or slow, but you sure seem to have to work a lot harder to go faster! Well that's because work is force times distance:

$$W = F \cdot s$$

and when you go faster you cover more distance in the same time. Now since work is just a realized form of energy, this formula says that force is energy per unit distance:

$$F = \frac{W}{s}$$

Engine efficiency. We define the efficiency e of the engine to be the amount of useful energy the engine can extract from a litre of gas. In terms of this, our overall model for gas consumption will have the form:

Total gas supplied $\quad z = \frac{av^3+bv}{e} \quad \left[\frac{\text{L}}{\text{h}} = \frac{\text{energy/h}}{\text{energy/L}}\right]$

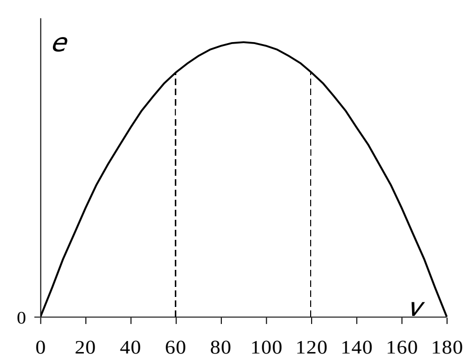
We just have to get hold of e. Now the efficiency of an engine, turns out to depend on the speed at which it is running. Most engines have an optimal running speed, at which the efficiency is a maximum, and the efficiency is lower if the engine speed is too high or too low. In fact that's where gears come in. The point of having gears in a car is to adjust the ratio of the speed of the wheels and the speed of the engine so that the engine can run close to its optimal speed at different driving speeds.

As a way of modeling this variation in e, assume that efficiency is a maximum at some "optimal" engine speed, and falls off slowly on either side. The simplest function around that will provide this is a parabola opening down, so let's use that. Then we simply have to decide where to put the vertex of the parabola and how "wide" to make it. Rather than work in terms of engine speed (rpm) we'll translate everything to the more familiar speed of the car—that's fine as long as we stay in a fixed gear.

Let's assume the gear is set so that the optimal speed is 90 km/h—thus the vertex of the parabola will be at v=90. I've no idea how wide the parabola should be (if I were keen, perhaps I'd write to a car manufacturer) but to get a mathematically simple e-formula, I'll suppose the parabola crosses the v-axis at the origin. Then the two roots will be v=0 and v=180, and the efficiency formula has the form:

$$e = kv(180 - v)$$

As a check on the reasonableness of this formula, let's see how it predicts that e should fall off as v departs from its optimal value of 90. At the right I tabulate and graph the formula.



v	e/k	%
90	2.5	100
80 or 100	8000	99
70 or 110	7700	95
60 or 120	7200	89

The results seem reasonable to me. With a 20 km/h departure, we lose 5% efficiency; and with a 30 km/h departure, we lose 11% efficiency. Actually that seems pretty good, but I guess that's what all them computers are for.

If we absorb the constant k into the parameters a and b, we get the formula:

$$z = \frac{av^3 + bv}{e} = \frac{av^3 + bv}{v(180 - v)} = \frac{av^2 + b}{180 - v}$$

4. Using calculus to solve the driving speed problem.

Determining the parameters

The graph we used for the driving speed problem was constructed using the gas model derived above:

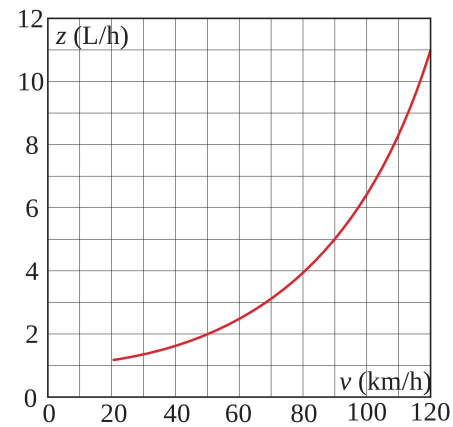
$$z = \frac{av^2 + b}{180 - v}$$

The parameters a and b were evaluated by taking a pair of data points obtained for a particular car. These are tabulated at the right. Putting these values into the general equation, we get the two equations:

v (km/h)	gas (L/h)
60	2.5
120	11

$$2.5 = \frac{a(60)^2 + b}{180 - 60}$$

$$11 = \frac{a(120)^2 + b}{180 - 120}$$



To solve these equations cross-multiply

$$300 = 3600a + b$$
$$660 = 14400a + b$$

These easily solve to give $a = 1/30$ and $b = 180$. We get

$$z = \frac{v^2 + 5400}{30(180 - v)}$$

This is, in fact, the equation I used to draw the z-v graph that we have been working with.

Minimizing $\frac{z}{v}$

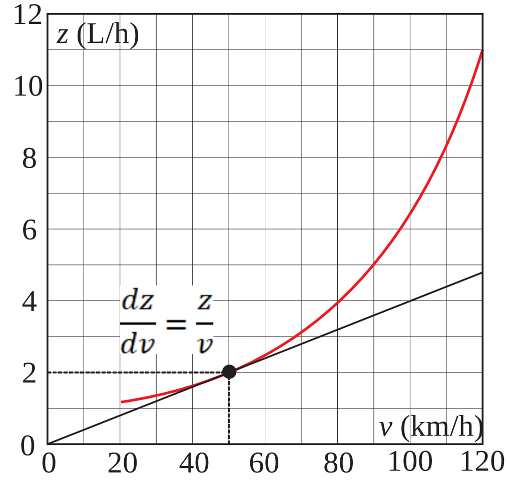
Problem 3. We begin with Problem 1, the case in which our only cost is the gas. In this case we want to minimize the amount of gas used per km and that's the quotient z/v.

But before we plug in the formula for z, it's worth continuing with the general expression. The minimum will be found where the derivative of z/v is zero.

We take the derivative with respect to v, keeping in mind that z is a function of v. Set it equal to zero.

$$\frac{d}{dv}\left(\frac{z}{v}\right) = \frac{\frac{dz}{dv}v - z\frac{dv}{dv}}{v^2} = 0$$

Now we solve for dz/dv:



$$\frac{dz}{dv}v - z = 0$$

$$\frac{dz}{dv} = \frac{z}{v}$$

That's an elegant condition. In fact it is exactly the condition for our geometric solution, that the slope of the tangent at the optimal point is equal to the slope of the line drawn to the point from the origin. Very satisfying.

Now we plug in the formula for z:

$$z = \frac{1}{30}\frac{v^2 + 5400}{180 - v}$$

And find its derivative

$$\frac{dz}{dv} = \frac{1}{30}\frac{2v(180 - v) - (v^2 + 5400)(-1)}{(180 - v)^2}$$

$$= \frac{1}{30}\frac{-v^2 + 360v + 5400}{(180 - v)^2}$$

Then the equation to be solved is:

$$\frac{dz}{dv}v = z$$

$$\frac{-v^2 + 360v + 5400}{(180 - v)^2} \cdot v = \frac{v^2 + 5400}{180 - v}$$

$$-v^2 + 360v + 5400)v = (v^2 + 5400)(180 - v)$$

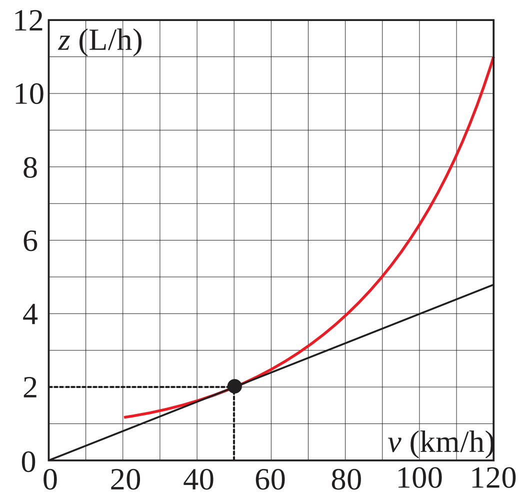
Note that the v^3 terms cancel.

$$(360 - 180)v^2 + (5400 + 5400)v - 5400 \cdot 180 = 0$$

$$180v^2 + 10800v - 5400 \cdot 180 = 0$$

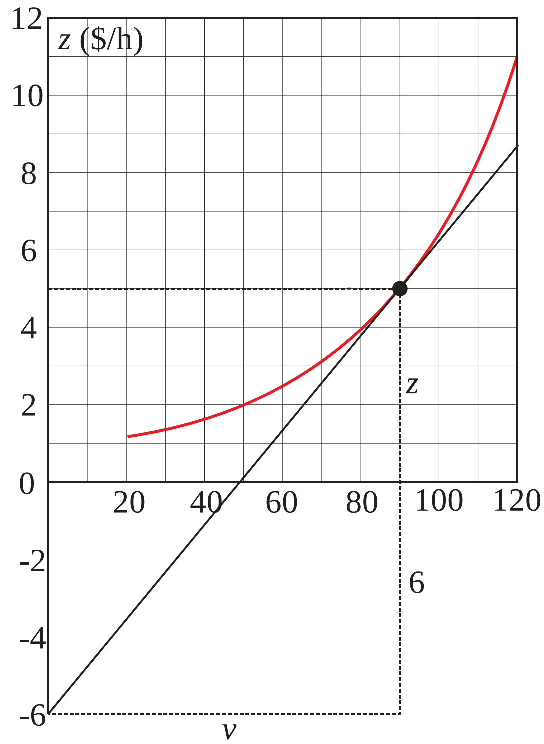
$$v^2 + 60v - 5400 = 0$$

$$v = \frac{-60 \pm \sqrt{3600 + 21600}}{2} \approx 49.4$$



This matches the answer we obtained geometrically.

Minimizing $\frac{z+6}{v}$



Problem 4. Now we move to Problem 2, the case in which we have both the cost of the gas ($1/L) and the wage ($6/h) of the driver. In this case we want to minimize the quotient $\frac{z+6}{v}$.

We take the derivative with respect to *v*, keeping in mind that *z* is a function of *v*. Set it equal to zero.

$$\frac{d}{dv}\left(\frac{z+6}{v}\right) = \frac{\frac{dz}{dv}v - (z+6)\frac{dv}{dv}}{v^2} = 0$$

Solve for dz/dv:

$$\frac{dz}{dv} = \frac{z+6}{v}$$

This is again the condition corresponding to our geometric solution. Write this as $\frac{dz}{dv}v = z + 6$ and adapting the Problem 1 solution:

$$\frac{1}{30}\frac{-v^2 + 360v + 5400}{(180-v)^2} \cdot v = \frac{1}{30}\frac{v^2 + 5400}{180 - v} + 6$$

Setting 180 = *k* and multiplying by 30, this is

I notice that there are many multiples of 180 in the expression so I simplify the algebra with the abbreviation *k*=180.

$$\frac{-v^2 + 2kv + 30k}{(k-v)^2} \cdot v = \frac{v^2 + 30k}{k - v} + k$$

$$= \frac{v^2 + 30k + k^2 - kv}{k - v} = \frac{v^2 - kv + k(30+k)}{k - v}$$

$$(-v^2 + 2kv + 30k)v = (v^2 - kv + k(30+k))(k - v)$$

Again the v^3 terms cancel. But this time the v^2 terms also cancel––there are 2*k* of them on either side. So we are left with the *v*-terms and the constant term.

$$30kv = (-k^2 - 30k - k^2)v + k^2(30+k)$$

Cancel *k*:

$$30v = -(2k + 30)v + k(30+k)$$

$$(2k + 60)v = k(30+k)$$

Cancel (30+*k*):

$$2v = k$$

$$v = \frac{k}{2} = 90$$

It is truly miraculous how nicely this calculation works out.

and this matches the answer we obtained geometrically.