Exponential dice.

Students work quite a bit with exponential growth and decay but do they ever see it at work? Do they ever actually construct an exponential process?

One of my valued teaching resources is my box of 100 dice. If you don't have one, it's well worth the investment. Just make sure you remind the students to bring them back each time! I get good results by dividing the box in half and have each group of students work with a set of 50 dice.

Here's the game. Roll the 50 dice. Remove and count all those that have come up six. Take up and roll the remaining dice. Again, remove and count all the sixes, and repeat. Keep going until there are no more dice left. Keep track of the numbers.

The table at the right records the results of 10 such experiments. The very first column keeps track of the number of rolls *n*, and each of the other ten columns represents a run of the experiment. The columns tabulate the number *N* still on the table after *n* rolls.

Look at the first trial E1. The first roll produced 7 sixes leaving *N* = 43 on the table. The second roll produced 10 sixes taking us down to *N* = 33. Then on the next roll only 1 six etc. Overall 1/6th of the dice should be sixes but there's lots of variation.

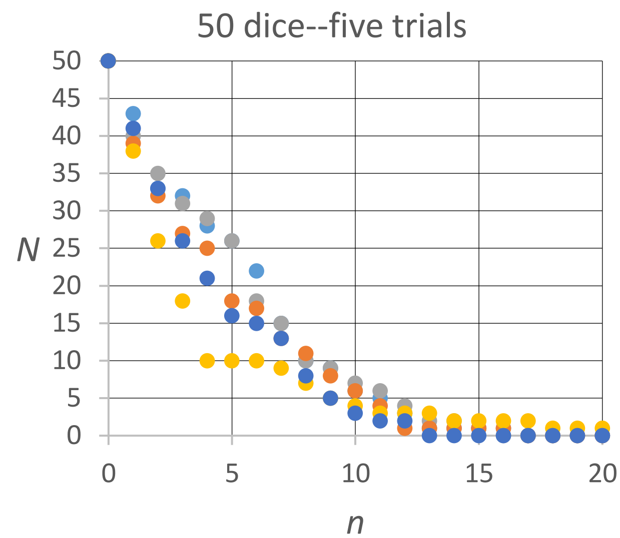
The longest of the trials is E4. The first roll produced 12 sixes leaving *N* = 38 on the table. The second roll also produced 12 sixes, so the numbers dropped unusually quickly at the beginning. But when we finally got down to 1 six (after roll 18) it hung on with no six until roll 32.

There are a few simple experiments around that produce exponential decay. One of my favorites is the tire pressure experiment, partly because it involves some unexpected modeling and partly because it gives an astonishingly good fit to the exponential decay curve.

This experiment is analogous to the tire model but with more variation, essentially because there are many fewer dice than there are molecules of air in the tire. And it's much more "hands on."

n	**E1**	**E2**	**E3**	**E4**	**E5**	**E6**	**E7**	**E8**	**E9**	**E10**
0	50	50	50	50	50	50	50	50	50	50
1	43	40	39	38	41	47	39	43	41	42
2	33	35	32	26	33	40	35	37	33	38
3	32	31	27	18	26	33	27	30	28	35
4	28	29	25	10	21	24	22	24	24	30
5	26	26	18	10	16	21	20	16	20	26
6	22	18	17	10	15	16	17	14	20	23
7	15	15	13	9	13	10	14	11	15	21
8	10	10	11	7	8	9	11	8	8	18
9	9	9	8	5	5	7	9	5	7	15
10	6	7	6	4	3	4	7	4	6	14
11	5	6	4	3	2	3	4	4	6	12
12	3	4	1	3	2	3	4	3	5	9
13	1	2	1	3	0	1	2	2	4	8
14	1	2	1	2	0	1	2	1	3	7
15	1	1	1	2	0	0	1	1	3	5
16	1	0	1	2	0	0	1	1	2	4
17	0	0	0	2	0	0	0	1	1	4
18	0	0	0	1	0	0	0	1	1	3
19	0	0	0	1	0	0	0	1	1	1
20	0	0	0	1	0	0	0	1	1	1
21	0	0	0	1	0	0	0	1	1	0
22	0	0	0	1	0	0	0	1	0	0
23	0	0	0	1	0	0	0	1	0	0
24	0	0	0	1	0	0	0	1	0	0
25	0	0	0	1	0	0	0	1	0	0
26	0	0	0	1	0	0	0	1	0	0
27	0	0	0	1	0	0	0	0	0	0
28	0	0	0	1	0	0	0	0	0	0
29	0	0	0	1	0	0	0	0	0	0
30	0	0	0	1	0	0	0	0	0	0
31	0	0	0	1	0	0	0	0	0	0
32	0	0	0	1	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0

The graph at the right plots the data from the first five trials for n = 0 to 20. It gives us a visual sense of the kind of variation we can get from trial to trial.



What kind of function are these data supposed to fit? What's the "theoretical" size of the number N of dice remaining at time n?

This is an important question and its answer involves a few key ideas so it's worth treating it properly.

For a student who has worked with these ideas before, the answer might seem simple enough. At each roll, the dice that come up six are eliminated, and that happens with probability 1/6. If there were a large number of dice, we'd expect a sixth of them to disappear with each roll, leaving 5/6 of them on the table. Thus, with each roll, we expect the dice population to be multiplied by 5/6. Starting with 50 dice, the number remaining after n rolls should be

$$N = 50\left(\frac{5}{6}\right)^n.$$

Here's a more leisurely stroll through this argument. Let $N = N(n)$ be the number of dice remaining at time n. To get a theoretical formula for N we use recursive thinking and ask how each value of N is obtained from the previous value.

Let's start at the beginning, $N(0) = 50$. How do we find $N(1)$? Well on the first roll we expect a sixth of these 50 dice to be removed, so that:

$$\begin{aligned} N(1) &= 50 - 50\left(\frac{1}{6}\right) \\ &= 50\left(1 - \frac{1}{6}\right) \\ &= 50\left(\frac{5}{6}\right). \end{aligned}$$

In the Ontario Grade 11 curriculum there is a unit on sequences and recursive relationships. If that unit was given the time that it deserves, the students should have seen this argument before. If it wasn't, well there's no time like the present!

For my money, a hugely important idea in the grade 11 curriculum is the idea of recursive thinking, and that should be a main theme for both the strands of discrete functions and exponential functions.

Now use the same argument to find $N(2)$. This time our starting number is not 50 but is $N(1)$:

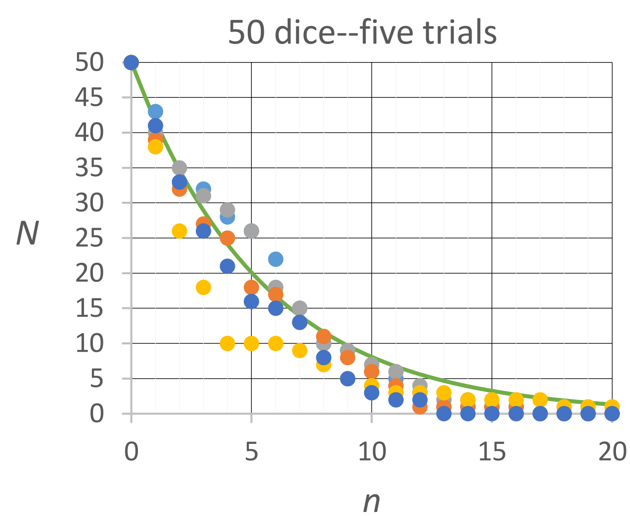
$$N(2) = N(1)\left(\frac{5}{6}\right)$$
$$= 50\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$$
$$= 50\left(\frac{5}{6}\right)^2$$
$$= N(0)\left(\frac{5}{6}\right)^2.$$

By now most students are quite familiar with this argument and would write the final formula down right away. But it doesn't hurt to be reminded how it goes.

By now the procedure and the formula are clear––each time we multiply by 5/6. Thus:

$$\boxed{N(n) = N(0)\left(\frac{5}{6}\right)^n}$$

At the right, the theoretical exponential curve is plotted along with the data from the first five trials. The curve seem to give a reasonable fit to the data, though there's lots of scatter.

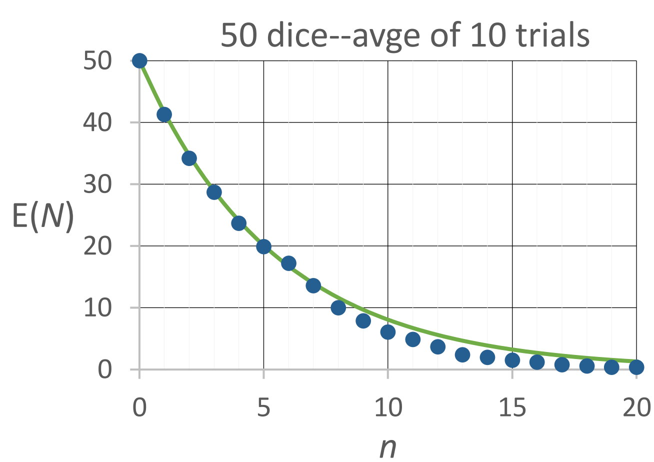


How to reduce the scatter.
A general principle is that the average of a large number of trials has much smaller "variance" than a set of single trials. For example if you throw 10 darts hoping to hit the bulls-eye you'll get a scatter around the centre. The average of those 10 throws will tend to be much closer to the centre than any typical throw.

At the right I have plotted the average of all 10 of our experiments, that is, at each time n, I took the average of the entries in the nth row. Along with that I plotted the theoretical curve

$$N = 50\left(\frac{5}{6}\right)^n$$

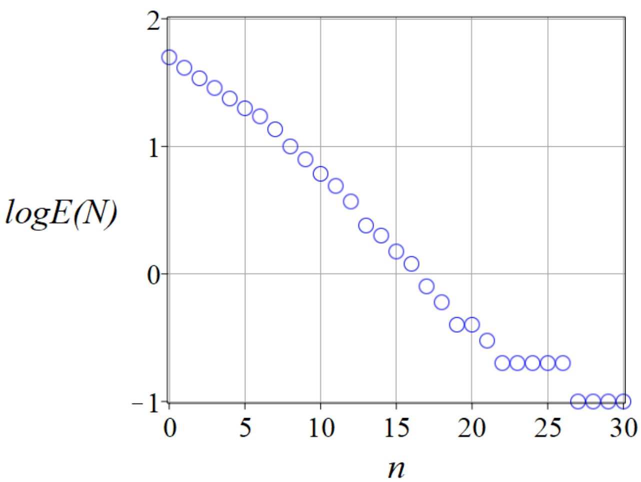
It looks like a pretty good fit.



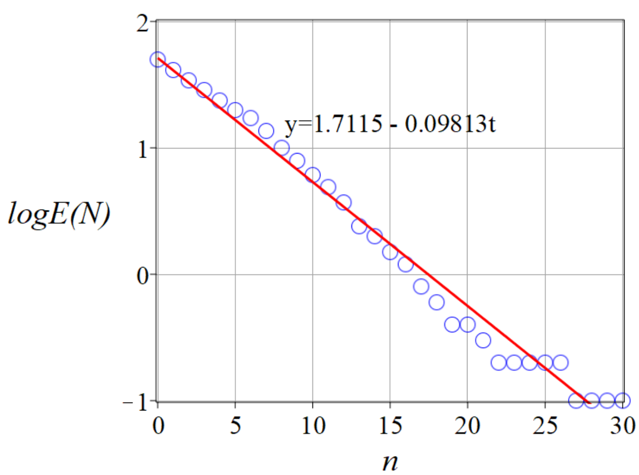
Another way to think of this average is as the data from a single experiment starting with 500 dice. We'd get the graph for that simply by multiplying the vertical scale by 10.

Linearizing the data.
But the real test of whether the curve is exponential is to plot its logarithm and see if we get a reasonably straight line. An interesting introduction to the logarithm is found in the tire-pressure model.

Our goal is to check whether the E(N) data is exponential, and to do that we have calculated its logarithm (column 3 at the right). The graph is found below. Note that here we are plotting up to n = 30.



The points appear to lie in a pretty good straight line and in the graph below the "best-fit" line for the full data set has been drawn.



$$\log(N) = 1.7115 - 0.09813t$$

$$N = 10^{1.7115-0.09813n}$$

$$= 10^{1.7115}(10^{-0.09813})^n$$

$$= 51.46(0.7978)^n.$$

n	E(N)	logE(N)
0	50.0	1.70
1	41.3	1.62
2	34.2	1.53
3	28.7	1.46
4	23.7	1.37
5	19.9	1.30
6	17.2	1.24
7	13.6	1.13
8	10.0	1.00
9	7.9	0.90
10	6.1	0.79
11	4.9	0.69
12	3.7	0.57
13	2.4	0.38
14	2	0.30
15	1.5	0.18
16	1.2	0.08
17	0.8	–0.10
18	0.6	–0.22
19	0.4	–0.40
20	0.4	–0.40
21	0.3	-0.52
22	0.2	-0.70
23	0.2	-0.70
24	0.2	-0.70
25	0.2	-0.70
26	0.2	-0.70
27	0.1	-1.00
28	0.1	-1.00
29	0.1	-1.00
30	0.1	-1.00

Note that the log(y) is negative when y is less than 1. Indeed
$10^0 = 1$ and
$10^x > 1$ for x positive
$10^x < 1$ for x negative

That base number 0.7978 is the approximation to theoretical value 5/6 given by our data. It's a bit small (5/6 = 0.833) but on the other hand the estimate 51.46 of the starting number of dice is a bit large.

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