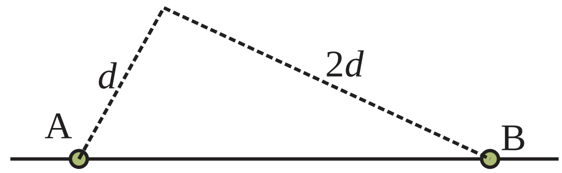
Grain elevators

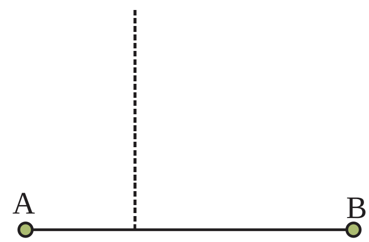
I am wandering on the prairie grasslands and I notice two tall grain elevators in the distance that appear to me to be exactly the same height. Now I know those elevators. I call them A and B, and it is known that B is exactly twice as tall as A. To amuse myself I decide to try to walk towards the elevators in such a way that they always appear to me to be the same height. What can you say about the shape of the path I will have to follow?

Well since B is twice the height of A, every point on my path will need to be twice as far from B as from A.

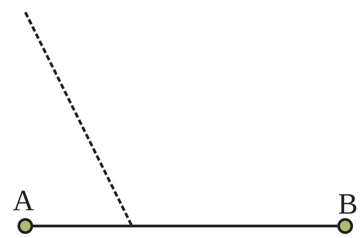


That’s pretty much all I give the students, and right away they begin to make sketches.

One student tries a right bisector.



Another thinks that maybe the path should be slanted.

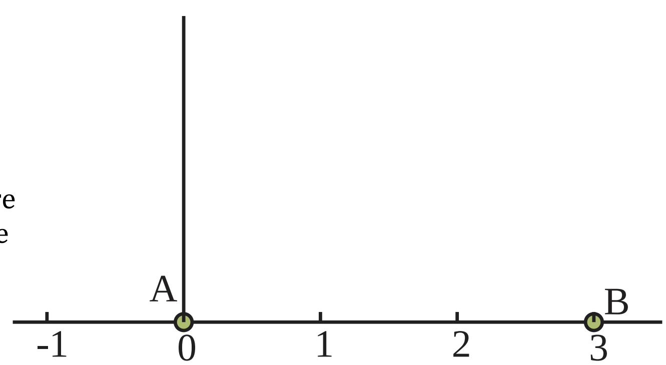


But simple measurements quickly show that these both must be wrong

Other guesses are made but students soon feel the need for a systematic approach. “How do we tackle this problem?” I suggest analytic geometry.

We need to set up a coordinate system. I let them think about this for a bit and then in order to get some uniformity in the class I instruct them to put the origin at A and draw the *x*-axis so that it passes through B.

How should we choose the scale? Think about where the path will eventually cross the *x*-axis—it will have to be twice as far from B as from A (the origin). Let that point be (0, 1) so that B will be at (0, 3).



To get the class doing a preliminary calculation I ask them to find the point where my path will intersect the y-axis.

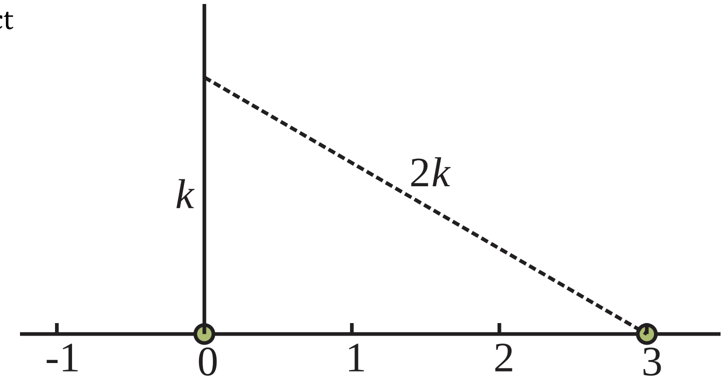
Letting the point be k units above the origin, a Pythagorean argument gives $k = \sqrt{3}$:

$$k^2 + 3^2 = (2k)^2$$

$$k^2 + 3^2 = 4k^2$$

$$9 = 3k^2$$

$$k^2 = 3$$



Now we have the two axes intersections of the path, one at (1, 0) and the other at $(0, \sqrt{3})$. What is the curve that joins them?

We let (x, y) be an arbitrary point on this path. The condition is that it be twice as far from B(0, 3) as from A(0, 0).

$$|(x, y) - (0,3)| = 2|(x, y) - (0,0)|$$

$$|(x, y) - (0,3)|^2 = 4|(x, y) - (0,0)|^2$$

$$(x^2 + (y - 3)^2 = 4(x^2 + y^2)$$

$$x^2 + (y^2 - 6y + 9) = 4x^2 + 4y^2$$

$$0 = 3x^2 + 3y^2 + 6y - 9$$

$$x^2 + y^2 + 2y - 3 = 0$$

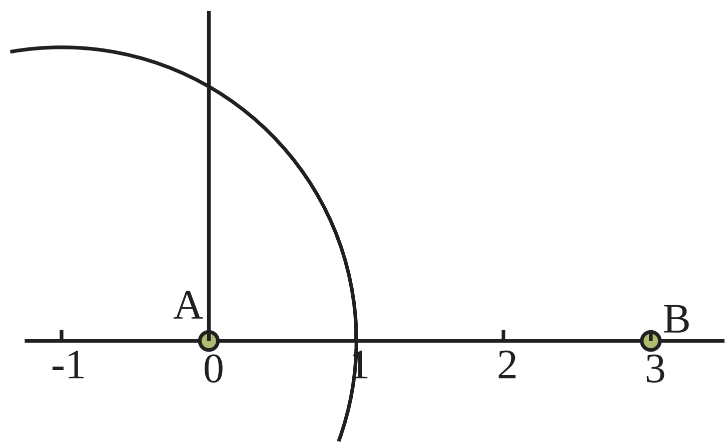
$$x^2 + y^2 + 2y = 3$$

$$x^2 + (y^2 + 2y + 1) = 4$$

$$x^2 + (y + 1)^2 = 4$$

$$|(x, y) - (0, -1)|^2 = 4$$

$$|(x, y) - (0, -1)| = 2$$



This tells us that the point (x, y) on the path is always distance 2 from the point $(0, -1)$. The path is a circle of radius 2 centred at the point $(0, -1)$.

As a check verify that the y-intercept $(0, \sqrt{3})$ is at distance 2 from the centre $(-1,0)$ of the circle.