Grade 12. Inverse quartic

An important notion in Grade 12 is that of the inverse of a function, but the examples that the students are given to work with are often not so interesting. Understanding come from examples that have a rich enough structure. I have been happy with the level of complexity that this example provides.

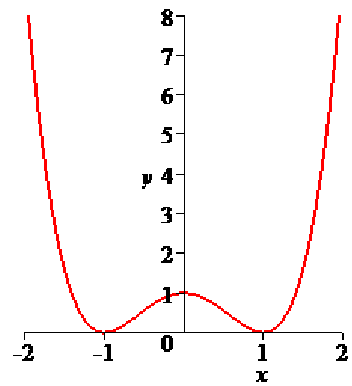
The graph of the quartic (degree 4) polynomial

$$y = (x^2 - 1)^2$$

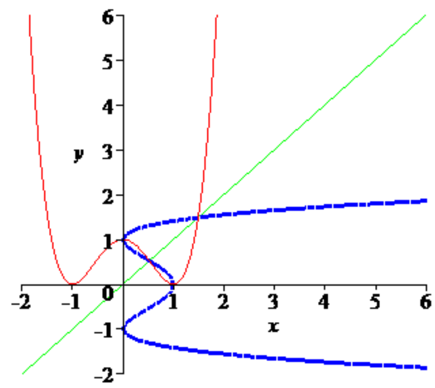
is drawn at the right. If we factor $x^2 - 1$ we can write y in factored form:

$$(x-1)^2(x+1)^2$$

and this clearly displays the double roots at $x = 1$ and $x = -1$. Take a few moments to decide that the graph looks exactly the way you'd expect it to.

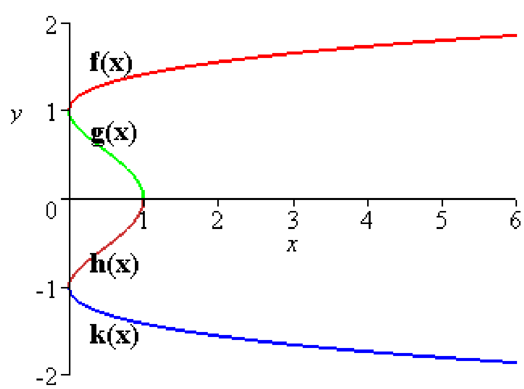


At the right is drawn the inverse graph, which is the reflection of the original in the line $y = x$. Points (a, b) on the original graph correspond to points (b, a) on the inverse.



Although $F(x) = (x^2 - 1)^2$ is a function, its inverse is not. For a graph to be a function, each value of x must give a unique value of y. Geometrically, that means that each vertical line can intersect the graph at most once. For the graph of the inverse, values of x between 0 and 1 give *four* corresponding values of y. Two of these values are positive, while the other two are negative. Since the graph is symmetrical about the x-axis, the negative values are the opposites of the positive values.

However, if we break the graph into the four pieces shown in color at the right, each individual piece is the graph of a function. We can call these four pieces the "functional pieces" and they are labeled from top to bottom as $f(x)$, $g(x)$, $h(x)$, and $k(x)$.



Your job is to find equations for each of these four functional pieces. In each case, pay attention to the domain of the function and check that in each case, the equation and the graph give us the same answer.

Solution.
The original equation is $y = (x^2 - 1)^2$. To find a corresponding equation for the inverse, we can interchange x and y and attempt to solve for y.

$$x = (y^2 - 1)^2$$

Solve for y. Remember that by definition, the square roots of a^2 are $\pm a$.

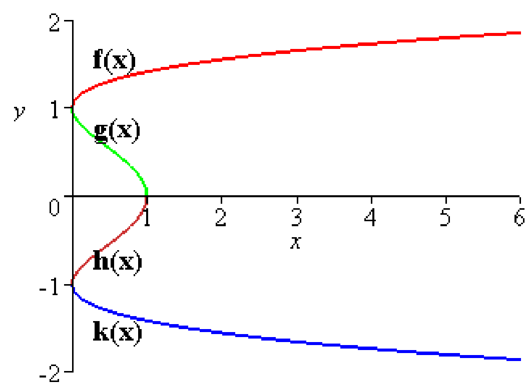
$$\pm\sqrt{x} = (y^2 - 1)$$

$$\pm\sqrt{x} + 1 = y^2$$

$$y^2 = 1 \pm \sqrt{x}$$

$$y = \pm\sqrt{1 \pm \sqrt{x}}$$

This is a general expression for the inverse. In fact, It really provides *four* expressions––there are two ± signs and we have two choices for each, for a total of four choices. These four expressions correspond to the four functional pieces of the inverse function.



How do we match the expressions with their corresponding functional pieces? That's a good problem to give the class. And given that, why do the domains make sense––that f and k are defined for all $x \geq 0$, but g and h only make sense for $0 \leq x \leq 1$?

Here's one way to argue. First f is the largest so it uses both plus signs. Then k is clearly the negative of f so the first + sign becomes a minus. To distinguish the remaining two, note that $g(x)$ is positive so the leading sign must be + and then h is what remains, and as a check note that h is indeed $-g$.

function	*domain*
$f(x) = +\sqrt{1 + \sqrt{x}}$	$x \geq 0$
$g(x) = +\sqrt{1 - \sqrt{x}}$	$0 \leq x \leq 1$
$h(x) = -\sqrt{1 - \sqrt{x}}$	$0 \leq x \leq 1$
$k(x) = -\sqrt{1 + \sqrt{x}}$	$x \geq 0$

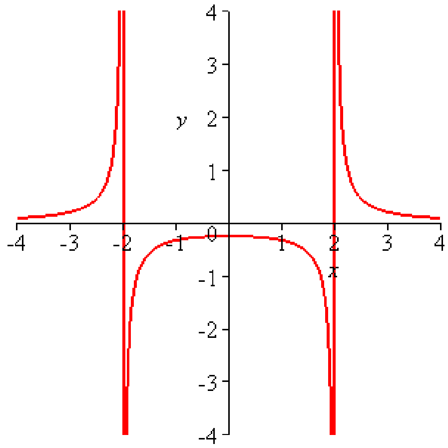
When working with functions, take every chance you get to make connections between the information provided by the graph and the information provided by the equation. The interaction between geometry and algebra provides insights that may not be readily apparent by considering the graph and equation separately

Problems.

1. Graph the function $y = x^2 - 4$. Determine its inverse both graphically and algebraically. Determine the equations for the functional pieces of the inverse graph and match each with the corresponding component of the graph. Specify the domain and range of each functional piece.

2. Repeat problem 1, this time using the function $y = x^2 - 2x$.

3. The function $y = \frac{1}{x^2 - 4}$ is graphed at the right.



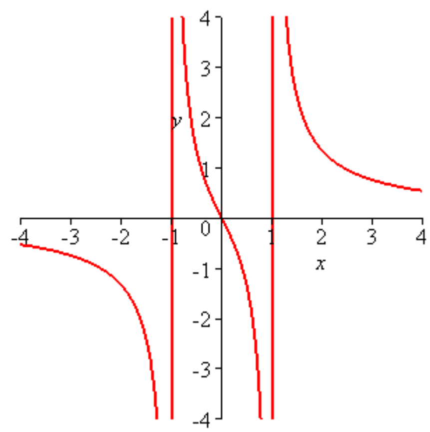
Distinguishing features are its bilateral symmetry about the y-axis, and the vertical asymptotes at $x = \pm 2$. For $x > 2$, the function decreases approaching zero for large x. For x between 0 and 2, the function is negative and to think about how it should look, write it as $y = -\frac{1}{4 - x^2}$ and think about the behaviour of $4 - x^2$. Draw the inverse graph and determine the equations for its functional pieces matching each with the corresponding component of the graph. Specify the domain and range of each functional piece.

4. Repeat problem 1, this time using the function $y = \frac{1}{(x^2 - 4)^2}$ Begin by drawing its graph (it is the square of the above graph).

5. (a) Draw the graph of $y = (x^2 - 1)^2$ of Example 1 but translated 1 unit to the right.

(b) Determine the equation of this translated graph.
(c) Find the inverse of the translated function, both graphically and algebraically, and identify the functional pieces of the inverse.
(d) Are the functional pieces identical to those of the inverse of $y = (x^2 - 1)^2$ shifted 1 unit up? Explain why or why not.

6. The function $y = \frac{2x}{x^2 - 1}$ is graphed at the right. Draw the inverse graph and determine the equations for its functional pieces matching each with the corresponding component of the graph. Specify the domain and range of each functional piece.



7. By any method, make or find a careful drawing of the graph of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. Note that the expression on the left is defined for both negative and positive values of *x* and *y*.
a) There are many types of symmetry exhibited by the graph. Describe them all.
b) Solve the equation for *y* as a function of *x*.
c) Determine the inverse both graphically and algebraically. Identify all functional pieces of the inverse.
d) State the domain of each functional piece and determine its equation.