Missteaks

> This unit has a bit of fun and might work in a Grade 9 class. It gives the students practice in working with quotient expressions, solving for one variable in terms of the others, and making simple arguments.

Observe that

$$\frac{64}{16} = \frac{4}{1}$$

We get the left-hand side by canceling the 6's. Of course that's not something we are "allowed" to do. But it gets the right answer anyway, and we ask whether there are other examples like that.

I believe that I first saw this example many years ago in an article with title *Mathematical Missteaks* but I can't seem to find it anymore. Anyway, such examples are not only amusing, they can lead to interesting questions. I provide a few here that will give the student an exercise in simple algebra. I will come back to the opening example in Ex. 3.

Example 1.
Consider that

$$4\tfrac{4}{3} = 4 \cdot \tfrac{4}{3}.$$

If the notation confuses you, I would say it as "4 *and* $\frac{4}{3}$ equals 4 *times* $\frac{4}{3}$." Are there other examples like this?

Solution. We can write this as

$$4 + \frac{4}{3} = 4 \cdot \frac{4}{3}$$

so we are looking for an integer *a* and a fraction *b* for which

$$a + b = a \cdot b$$

Solve for *b*:

$$b(1 - a) = -a$$

$$b = \frac{a}{a - 1}$$

That gives us the *structure* of the original example, the 4 can be any integer and the 3 is simply one less than the 4. To find other cases we go through the integer values of *a.* The first cases, *a* = 1 and *a* = 2 don't give us examples but

a = 3 $\quad 3\tfrac{3}{2} = 3 \cdot \tfrac{3}{2}$

a = 4 $\quad 4\tfrac{4}{3} = 4 \cdot \tfrac{4}{3}$

a = 5 $\quad 5\tfrac{5}{4} = 5 \cdot \tfrac{5}{4}$

a = 6 $\quad 6\tfrac{6}{5} = 6 \cdot \tfrac{6}{5}$

etc.

Example 2.

Consider that $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$.

Again, if the notation confuses you, this says that the square root of 3 and $\frac{3}{8}$ equals 3 times the square root of $\frac{3}{8}$. Are there other examples like this?

Solution. We can write this as

$$\sqrt{3 + \frac{3}{8}} = 3\sqrt{\frac{3}{8}}$$

so we are looking for an integer *a* and a fraction *b* for which

$$\sqrt{a+b} = a\sqrt{b}.$$

Square both sides: $a + b = a^2 b$

Solve for *b*: $b(1 - a^2) = -a$

$$b = \frac{a}{a^2 - 1}$$

That tells us that in the original example, the 3 can be any integer and the 8 is simply one less than 3^2. Let's go through the integer values of *a.* The first case, *a* = 1 doesn't give us examples but:

a = 2, *b* = 2/3: $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$

a = 3, *b* = 3/8: $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$

a = 4, *b* = 4/15: $\sqrt{4\frac{4}{15}} = 4\sqrt{\frac{4}{15}}$

etc.

Example 3.
Consider that

$$\frac{64}{16} = \frac{4}{1}$$

obtained by canceling the 6's. Can you find other examples?––let's restrict our search to 2-digit numbers.

This might be a good competitive problem. After giving the original example, let the class work in groups of 4 and see who can find the remaining examples first. Then students can get a chance at the end to compare strategies and observations.

Solution. The general form is

$$\frac{10a + b}{10c + a} = \frac{b}{c}$$

Cross-multiply: $(10a + b)c = (10c + a)b$

$$10ac + bc = 10cb + ab$$

Solve for a: $a(10c - b) = 9bc$

$$a = \frac{9bc}{10c - b}$$

And a, b and c must all be integers between 1 and 9.

Now it's really just a question of trying out all the possibilities of the expression on the right for different values of b and c. There are some observations that will cut down the possibilities.

First we can ignore the case $b = c$. Indeed if $b = c$ then $a = \frac{9b^2}{9b} = b$ and so $a = b = c$ giving us uninteresting examples such as $\frac{22}{22} = \frac{2}{2}$. Secondly we should always keep in mind that the denominator $10c–b$ must always divide evenly into the numerator and we encounter lots of cases where we can see right away that this won't happen (i.e when we get a large prime number or factor in the denominator).

A reasonable process is to consider, one at a time, the different possibilities for c.

$c = 1$: $a = \frac{9b}{10-b}$

We get two possibilities, $b = 4$ and $b = 5$, giving us $a = 6$ and $a = 9$:

and we get the example we started with: $\frac{64}{16} = \frac{4}{1}$

and a new one: $\frac{95}{19} = \frac{5}{1}$

$c = 2$: $a = \frac{18b}{20-b}$

and $b = 5$, $a = 6$ is the only result: $\frac{65}{26} = \frac{5}{2}$.

It turns out that there is only one other:

$c = 4$: $a = \frac{36b}{40-b}$

and $b = 8$, $a = 9$ is the only possibility, giving us: $\frac{98}{49} = \frac{8}{4}$.

I must say I can imagine that some groups might go for the brute-force no-math approach––try all possibilities. I wonder what strategy would win.

Example 4.
We've all noticed that $2^4 = 4^2$

but perhaps not explored this more deeply. But when we feast our eyes on:

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}$$

we start to wonder what's going on here. Is this the start of a sequence of examples, and if so, what's the next one?

There are many ways one might try to pick this problem apart. Certainly it's not at all clear that the first and second examples above are the first two terms of a natural sequence. What can we do?

Well the second example has on its own acompelling structure. One of the quantities 9/4 is the square of 3/2 and the other quantity 27/8 is the *cube* of 3/2. Taking *s* = 3/2 to be the *seed* of the equations we can write it as

$$(s^2)^{s^3} = (s^3)^{s^2}$$

Now let's work backwards and see if this equation "solves" for *s*. Being careful with the exponent laws:

$$s^{2s^3} = s^{3s^2}$$

$$2s^3 = 3s^2$$

$$2s = 3$$

$$s = \tfrac{3}{2}.$$

Very nice. Okay. This is a very nice structure for the second example. Does the first example $2^4 = 4^2$ have the same structure?

Yes it does. We reduce the indices by 1:

$$(s^1)^{s^2} = (s^2)^{s^1}$$

And this equation solves to give the seed *s* = 2 = 2/1.

Given all this, what will the third term be? Well perhaps it will have the seed *s* = 4/3 and equation:

$$(s^3)^{s^4} = (s^4)^{s^3}$$

Solving this indeed gives us *s* = 4/3. And that gives us our third example. We calculate $\left(\frac{4}{3}\right)^3 = \frac{64}{27}$ and $\left(\frac{4}{3}\right)^4 = \frac{256}{81}$ giving us

$$\left(\frac{64}{27}\right)^{\frac{256}{81}} = \left(\frac{256}{81}\right)^{\frac{64}{27}}$$

Wow.

This is a wonderful example. It is of a somewhat different kind but can still be regarded as a missteak in the sense that it's an example where doesn't matter if the student mixes up the base and the index.

The analysis for this example is more advanced, but it is a superb opportunity for probing the structure of a configuration.

This problem can also generate a good competition—put the students in small groups and see who can come up first with a new example.

Here's an instance of a basic exponent law: $(x^m)^n = x^{mn}$, but the complicated expressions might give students trouble.