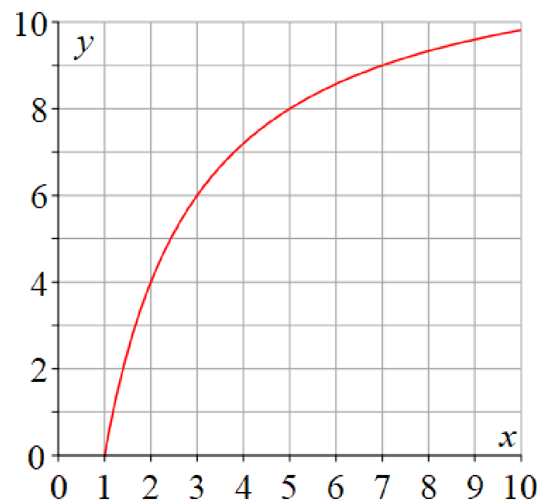
The magical money machine.

You have access to a magical money machine. You can put in any amount of money you want, between 0 and $10, and pull the big brass handle, and some "payoff" will come pouring out. Now this payoff depends on the amount you put in. If you feed it x dollars it will pay out $y = f(x)$, and the graph of $f(x)$ is given at the right.



For example, an input of $2 will yield a payoff of $4, whereas an input of $5 will yield a payoff of $8. Just to be clear, if you put in $5 and get $8 back, your profit or net gain is 8 – 5 = $3.

Note that pennies are allowed; for example it turns out that an input of $5.43 yields a payoff of $8.27. Of course the graph you are given does not have enough fine detail for you to obtain those exact numbers.

The problem is: how should you play to maximize your profit? What is your optimal input x?

After thinking about the problem a bit, you might well start to wonder what sort of access you have. Can you play only once? Can you play many times? How many times?

Well, I will give you two option and you can choose whichever you wish.

Option A. You are allowed to pull the handle a total of 100 times.
Option B. The total amount you feed into the machine cannot exceed $300.

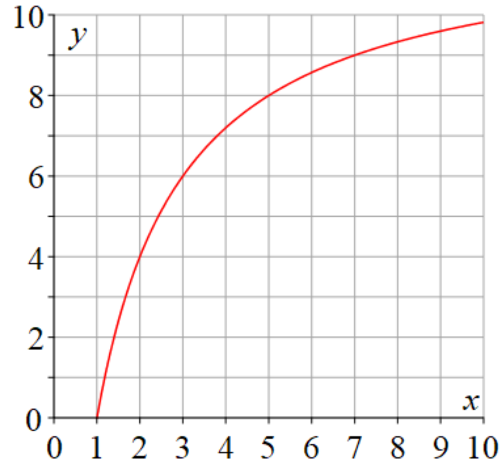
Which should you choose? In each case, what is the optimal value of x and what is your total profit?

To begin, we are going to tackle this problem using graphical arguments, and then we will use algebraic tools to verify our results.

Graphically this will give the student experience in what might be called qualitative thinking in mathematical modeling. Indeed the techniques we will use here are found in applications to the social and biological sciences in which we often don't have equations for our functions, but have only qualitative information about the shape of the graph.

Algebraically we will find a simple equation for the graph and then we will use some of the techniques we have already developed in our work with parabolas.

Graphical Analysis

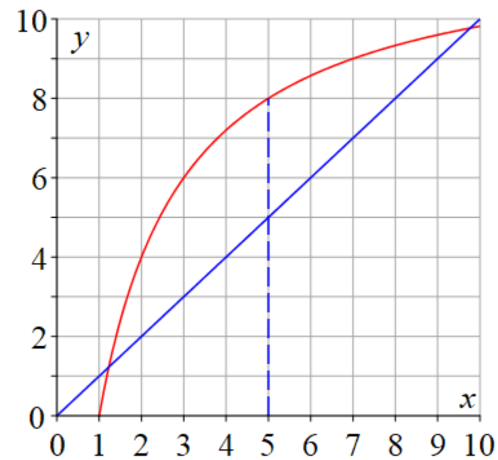


Option A. You are allowed to pull the handle a total of 100 times. How much should you put into the machine at each pull? What is your total profit?

If your input is *x* each time, your profit will be $f(x) - x$, the difference between your output and your input, and your total profit over 100 pulls will be

$$P(x) = 100[f(x) - x]$$

So we want to maximize $f(x) - x$.



How do we do this with the graph of *f*(*x*)? We take the two graphs

Output: $y = f(x)$

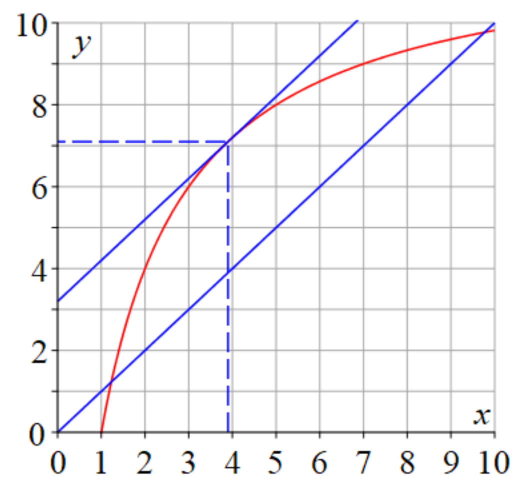
Input: $y = x$

and $f(x) - x$ is the vertical distance between them. Be sure you see why this is the case. It's good to take a particular value of *x*, say *x* = 5. The profit is

$$f(5) - 5 = 8 - 5 = 3$$

and that's the vertical distance between the graphs at *x* = 5.

Now for what *x* is that vertical distance a maximum?



The point at which this occurs is illustrated in the diagram at the right. It is the point at which the tangent to the *f*-graph is parallel to the line *y* = *x*. We get an optimal input very close to *x* =4, say at *x* = \$3.90. At this value of *x* we get a payoff of just over 7, say \$7.10, for a profit of 7.10 – 3.90 = \$3.20 per pull.

For 100 pulls, our profit is \$320.

Be sure you understand why the optimal *x*-value is given by the tangent construction. Since the tangent is parallel to the line *y*=*x* there is a constant vertical distance between the two lines. This is exactly the distance between the *f*-graph and the line *y*=*x* at *x* = 3.90, but since the *f*-graph lies *under* the tangent, it is greater than the distance between the curve and the line *y*=*x* at all other values of *x*.

Option B. You can pull the handle as many times as you want but the total amount of money you put in cannot exceed \$300. So the question now is how much should you put into the machine at each pull?

Okay. If you put in x at each pull, you'll get $300/x$ pulls, and so your total profit will be

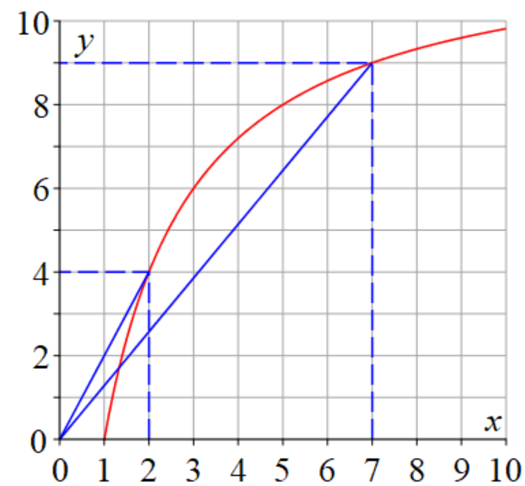
$$P(x) = \text{total output} - \text{total input}$$

$$P(x) = \frac{300}{x} f(x) - 300 = 300\left(\frac{f(x)}{x} - 1\right)$$

To maximize this we want to maximize

$$\frac{f(x)}{x}$$

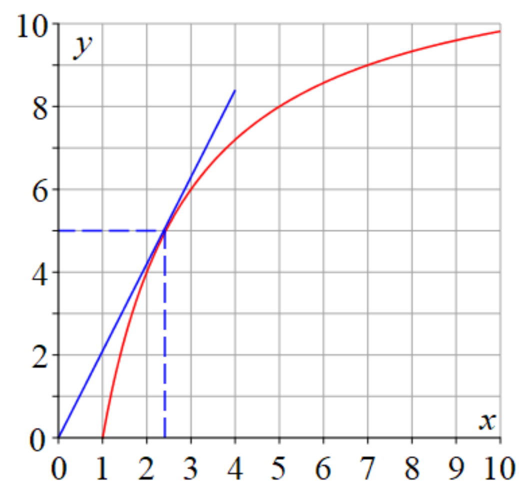
How do we do that graphically?



To get students started on this, it is reasonable to suggest that they think of the quotient as a slope, as a rise/run. For example, take $x = 2$. Can you find a "natural" line whose slope is

$$\frac{f(2)}{2} = \frac{4}{2}?$$

The answer is yes––it is the secant drawn from the origin to the point $(2, f(2))$ on the graph.



This secant is also drawn for $x = 7$. More generally, the line drawn from the origin to the point $(x, f(x))$ on the graph will always have

$$\text{slope} = \frac{f(x)}{x}$$

It is geometrically clear that this slope is a maximum at the point on the curve at which the line from the origin is tangent to the graph. This occurs at x close to 2.4 for an output of 5.0.

The answer is that you should put \$2.40 into the machine at each pull. Your profit from \$300 will be

$$P(x) = 300\left(\frac{f(x)}{x} - 1\right)$$

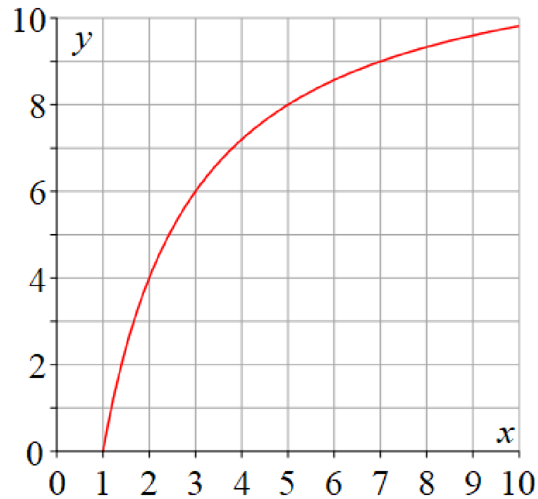
$$300\left(\frac{5.0}{2.4} - 1\right) = 325$$

The scores are in:

Option A: \$320
Option B. \$325

B is winner—but it was close.

Algebraic analysis—finding the *f*-equation



To begin we need the equation of the payoff curve. Here what I will tell you. The curve is constructed from a piece of the graph

$$y = \frac{k}{x}$$

for some value of $k > 0$. More precisely, if I cut out a piece of the graph $y = k/x$ and flip it upside-down and translate it, I could lay it exactly along the $f(x)$ graph. Let's use this to find a formula for $f(x)$.

upside down	$y = -\frac{k}{x}$
move up down	$y = b - \frac{k}{x}$
move right left	$y = b - \frac{k}{x-a}$

Our general form of $f(x)$ is:

$$f(x) = b - \frac{k}{x-a}$$

We have 3 unknowns a, b and k so we need 3 equations. We will get these from the first three data points on the graph:

$$x = 1, y = 0: \qquad b - \frac{k}{1-a} = 0$$

$$x = 2, y = 4: \qquad b - \frac{k}{2-a} = 4$$

$$x = 3, y = 6: \qquad b - \frac{k}{3-a} = 6$$

This gives us 3 equations in 3 unknowns. What we do now is eliminate one of the variables giving us 2 equations in 2 unknowns. An obvious candidate is b. Solve each equation for b and set the results equal:

$$\text{1st and 2nd:} \quad \frac{k}{1-a} = \frac{k}{2-a} + 4$$

$$\text{1st and 3rd:} \quad \frac{k}{1-a} = \frac{k}{3-a} + 6$$

Now we have two equations in the two unknowns k and a. We now solve for k:

$$\frac{k}{1-a} = \frac{k}{2-a} + 4$$

$$k\left(\frac{1}{1-a} - \frac{1}{2-a}\right) = 4$$

$$k\left(\frac{(2-a)-(1-a)}{(1-a)(2-a)}\right) = 4$$

$$k\left(\frac{1}{(1-a)(2-a)}\right) = 4$$

$$k = 4(1-a)(2-a)$$

$$\frac{k}{1-a} = \frac{k}{3-a} + 6$$

$$k\left(\frac{1}{1-a} - \frac{1}{3-a}\right) = 6$$

$$k\left(\frac{(3-a)-(1-a)}{(1-a)(3-a)}\right) = 6$$

$$k\left(\frac{2}{(1-a)(3-a)}\right) = 6$$

$$2k = 6(1-a)(3-a)$$

Eliminate k:

$$4(1-a)(2-a) = 3(1-a)(3-a)$$
$$4(2-a) = 3(3-a)$$
$$8-4a = 9-3a$$
$$-a = 1$$

We get that $a = -1$. Then

$$k = 4(1-a)(2-a) = 4(2)(3) = 24$$

and

$$b = \frac{k}{1-a} = \frac{24}{2} = 12$$

Then our payoff function is

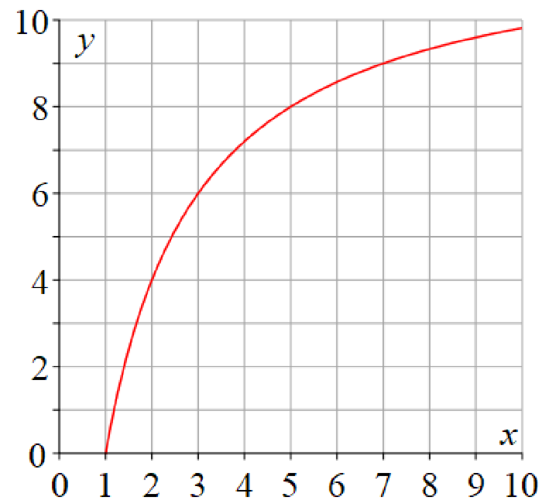
$$f(x) = b - \frac{k}{x-a}$$

$$= 12 - \frac{24}{x+1}$$

$$= \frac{12(x+1)-24}{x+1}$$

$$f(x) = 12\left(\frac{x-1}{x+1}\right)$$

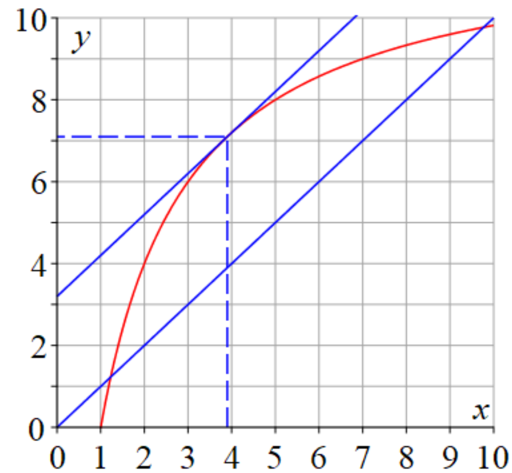
and this is our formula for $f(x)$. As a check we can verify that $f(5) = 8$ and $f(7) = 9$.



It is remarkable that this rational function delivers integer values at $x = 1, 2, 3, 5$ and 7.
Now we analyze the two options.

Algebraic analysis of each option

Option A. You are allowed to pull the handle a total of 100 times. How much should you put into the machine at each pull?



Our graphical analysis tells us that the optimal x is obtained where the payoff function has slope 1. To find this we take the family of all lines of slope 1:

$$y = x + b$$

and choose the line that intersects the payoff curve

$$f(x) = 12\left(\frac{x-1}{x+1}\right)$$

exactly once. Thus we want exactly one solution to the equation

$$x + b = 12\left(\frac{x-1}{x+1}\right)$$

$$(x+b)(x+1) = 12(x-1)$$

$$x^2 + (b+1)x + b = 12x - 12$$

$$x^2 + (b-11)x + (b+12) = 0$$

$$x = \frac{-(b-11) \pm \sqrt{(b-11)^2 - 4(b+12)}}{2}$$

We get exactly one solution when the discriminant is zero:

$$(b-11)^2 - 4(b+12) = 0$$

$$b^2 - 22b + 121 - 4b - 48 = 0$$

$$b^2 - 26b + 73 = 0$$

Solve for b:

$$b = \frac{26 \pm \sqrt{26^2 - 4 \cdot 73)}}{2}$$

The diagram tell us to use the negative sign. To simplify I take 4 out of the square root and divide top and bottom by 2

$$b = 13 - \sqrt{13^2 - 73)}$$

$$= 13 - \sqrt{169 - 73)}$$

$$= 13 - \sqrt{96} \approx 3.202$$

This is the y-intercept and that agrees with the diagram.

Interesting that we have used the quadratic formula twice in quick succession, first to solve for x and then to find the value of b that makes the discriminant zero.

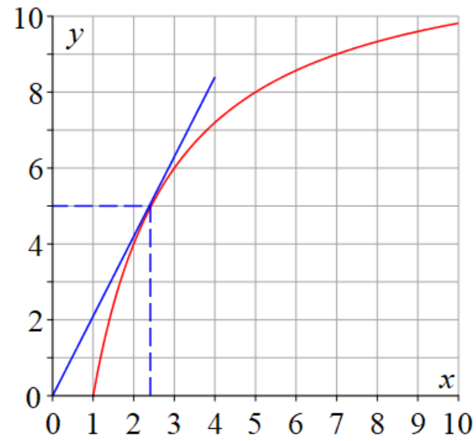
Now we calculate x. We use the x-formula above keeping in mind that the discriminant is zero:

$$x = \frac{-(b-11)}{2}$$

$$\approx \frac{11 - 3.2}{2} = 3.9$$

and this agrees with the diagram.

Option B. You can pull the handle as many times as you want but the total amount of money you put in cannot exceed \$300. How much should you put into the machine at each pull?



Our graphical analysis tells us that the optimal x is obtained where the line through the origin

$$y = mx$$

Is tangent to the payoff curve

$$f(x) = 12\left(\frac{x-1}{x+1}\right)$$

And that means there will be exactly one intersection. Thus we want exactly one solution to the equation

$$mx = 12\left(\frac{x-1}{x+1}\right)$$

$$mx(x+1) = 12(x-1)$$

$$mx^2 + mx = 12x - 12$$

$$mx^2 + (m-12)x + 12 = 0$$

$$x = \frac{-(m-12) \pm \sqrt{(m-12)^2 - 4m(12)}}{2m}$$

We get exactly one solution when the discriminant is zero:

$$(m-12)^2 - 48m = 0$$

$$m^2 - 24m + 144 - 48m = 0$$

$$m^2 - 72m + 144 = 0$$

Solve for b:

$$m = \frac{72 \pm \sqrt{72^2 - 4 \cdot 144)}}{2}$$

$$= 36 \pm \sqrt{36^2 - 144)}$$

Again we want the negative sign.

$$m = 36 - \sqrt{1152} \approx 2.059$$

Now we calculate x using the above x-formula keeping in mind that the discriminant is zero:

$$x = \frac{12-m}{2m} \approx \frac{6}{m} - \frac{1}{2} \approx 2.41$$

and this agrees with the diagram.

Problems.

Let's generalize options A and B. Suppose that in option A you are allowed to pull the handle P times and in option B you are allowed to put in a total of D dollars. Find condition on N and D for option B to be preferred. Use the results of our graphical analysis.

Solution. Our optimal x-values for both options were independent of the parameters N and D. In option A we had to maximize

$$100[f(x) - x] = N[f(x) - x]$$

and in option B we had to maximize

$$300\left(\frac{f(x)}{x} - 1\right) = D\left(\frac{f(x)}{x} - 1\right)$$

The first needs to maximize $f(x) - x$ and that happens at x = 3.2 and the second needs to maximize $(x)/x$ and that happens at x = 2.4. The condition for option B to be better is that

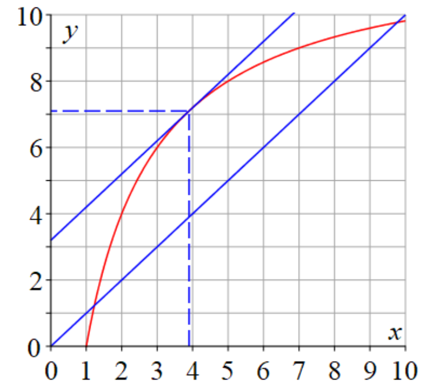
$f(x)/x$. The first occurred at x = 3.2 and the second occurred at x = 2.4. The payoff for the first is $3.2N$ and the payoff for the second is

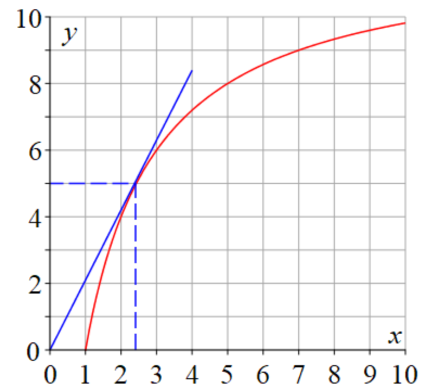
$$D\left(\frac{f(2.4)}{2.4} - 1\right) > N[f(3.9) - 3.9]$$

$$D\left(\frac{5}{2.4} - 1\right) > N[7.1 - 3.9]$$

$$D\left(\frac{2.6}{2.4}\right) > N[3.2]$$

$$\frac{D}{N} > \left(\frac{2.4 \times 3.2}{2.6}\right) = 2.95$$





Option C. Here's another option. As in Option B, you are allowed to put a total of $300 into the machine, however, every time you pull the handle you have to pay a $2 access fee. How much should you put into the machine at each pull?

We begin as in Option B. If you put in x at each pull, you'll get $\frac{300}{x}$ pulls. But your output on each pull is reduced by $2, so your total profit will be

$$P(x) = \text{total output} - \text{total input}$$

$$P(x) = \frac{300}{x}(f(x) - 2) - 300 = 300\left(\frac{f(x) - 2}{x} - 1\right)$$

To maximize this we want to maximize

$$\frac{f(x) - 2}{x}$$

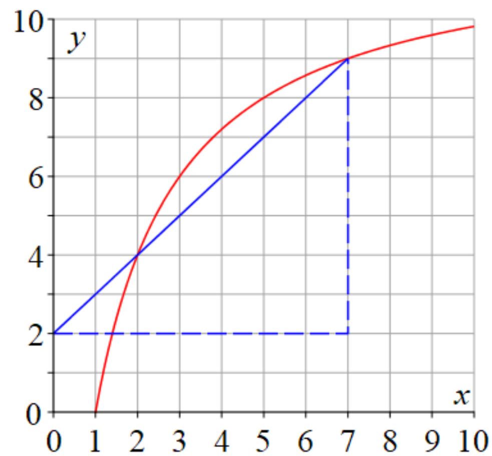
How do we do that graphically?

Well the same tangent trick should work if we replace the graph of $f(x)$ by the graph of $f(x) - 2$, that is, if we lower the graph of $f(x)$ by 2 units.

But we can in fact work with the diagram we already have by drawing the secants to the graph from a point 2 units above the origin. Just to check that, note that that the secant from that point to the point on the graph at $x = 7$ has slope

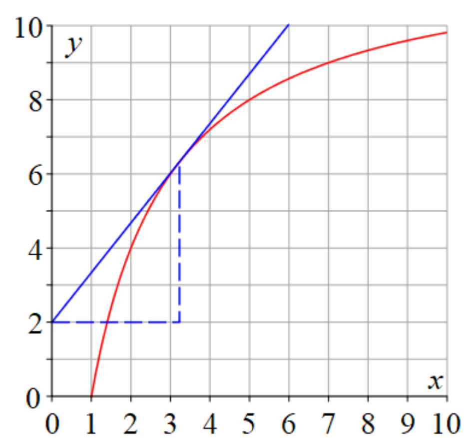
$$\frac{f(7) - 2}{7}$$

and in this case, that so happens to be 1.



Again the maximum slope occurs at the point at which the secant is tangent to the graph. This occurs at x close to 3.2 for an output of 6.3.

The answer is that you should put $3.20 into the machine at each pull.



Option D. Again there is a \$2 access fee every time you pull the handle and a \$300 total on the amount you spend but this total spending limit applies to the total money you put in to the machine *plus* the total paid in access fees. Thus the \$300 must be divided between the machine inputs and the total access fees. How much should you put into the machine at each pull?

Suppose you feed x into the machine at each pull. Then the total cost of each pull is x+2 and this is taken from the \$300. So the number of pulls you get is

$$\frac{300}{x+2}$$

Your output on each pull is $f(x)$ so your total profit will be

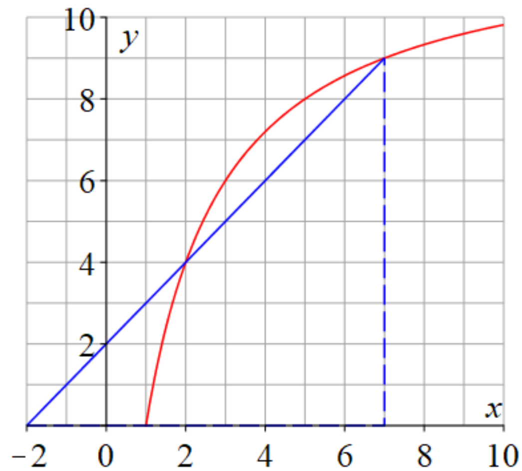
$$P(x) = \text{total output} - \text{total input}$$

$$P(x) = \frac{300}{x+2} f(x) - 300 = 300\left(\frac{f(x)}{x+2} - 1\right)$$

To maximize this we want to maximize

$$\frac{f(x)}{x+2}$$

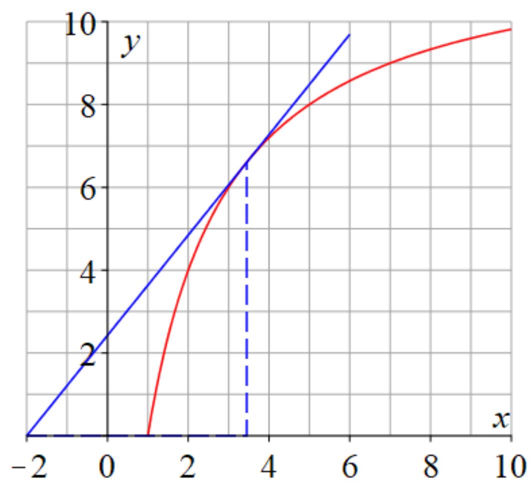
How do we do that graphically? Can we make our tangent trick work?



Well we want the "run" of our slope to be x+2 instead of x. One way we can do that is by drawing the secants from a point 2 units to the left of the origin. Just to see how this works, note that that the secant from that point to the point on the graph at x = 7 has slope

$$\frac{f(7)}{7+2}$$

and in this case, that also happens to be 1.



The maximum slope we can get from such secants is again at the point at which the secant is tangent to the graph. This occurs at x close to 3.5 for an output of 6.6.

The answer is that you should put \$3.50 into the machine at each pull.