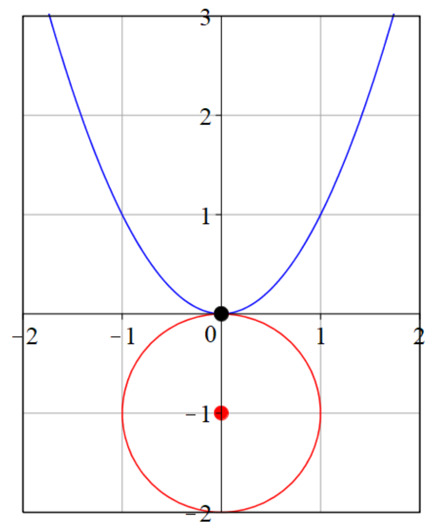
Parabola and circle



A circle of radius 1 with centre (0, −1) sits below the parabola

$$y = x^2$$

and is tangent to it at the origin. If we let the circle move up the y-axis, that tangent point will become a pair of intersections.

If the circle keeps moving up the number of intersections will continue to change until finally the circle will be entirely above the parabola with no further intersections.

Your job is to tell the story of the changing number of intersections in terms of the centre $(0, t)$ of the circle, and to construct an animation that illustrates this behaviour.

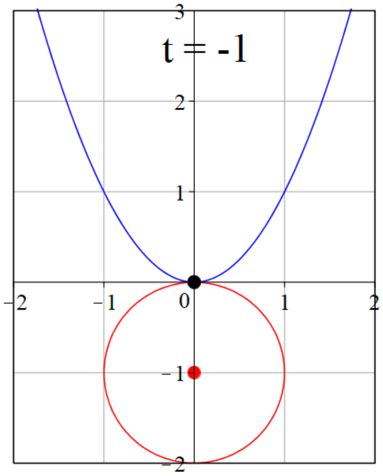
For the first stage, we want the students to use only their graphical intuition and see how much they can do. Or play with the Jupyter Notebook.

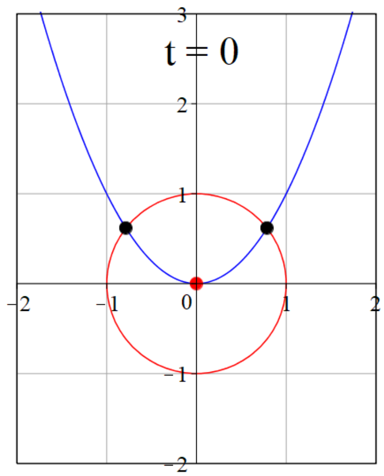
In the second stage they use the algebra to confirm their conclusions.

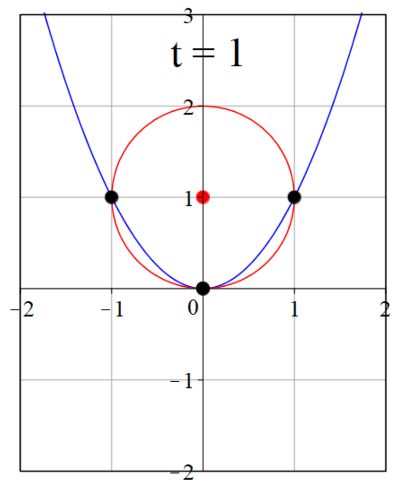
Graphical reasoning.

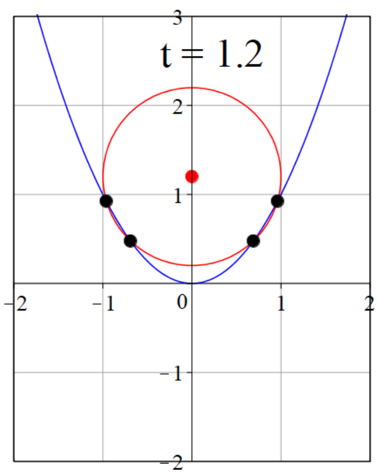
Here's what the students might come up with. A table of the sequence of numbers of intersections together with a sequence of matching diagrams. The one value that is hard to come up with we have called t^*. Though with the use of the notebook the students might guess its value to be $t^* = 1.25$.

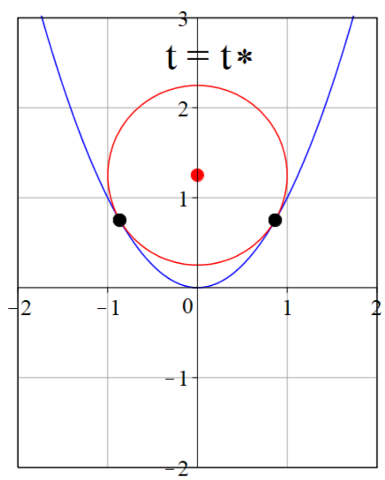
condition	#intersections
$t = -1$	1
$-1 < t < 1$	2
$t = 1$	3
$1 < t < t^*$	4
$t = t^*$	2
$t > t^*$	0

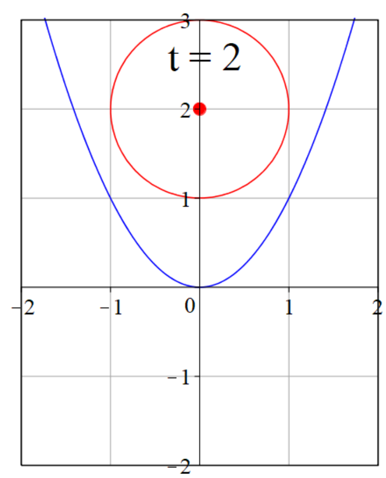












Algebraic reasoning
The graphical work will have given the students a pretty good idea of what t^* might be. In fact with some good manipulatives, some of them will have conjectured the value of $t^* = 5/4$. But it's an important exercise for them to check this out with the algebra. And if any of them want to build an animation they will certainly need to go there.

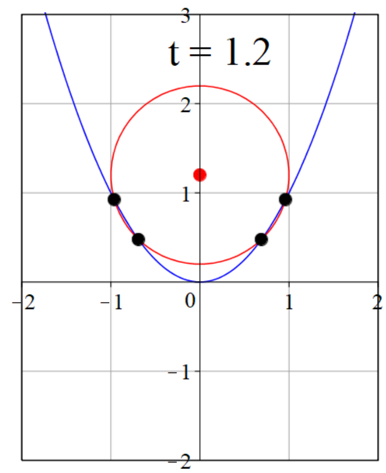
The circle and the parabola will intersect at points (x, y) that are on both curves, and thus satisfy both equations. We have the equation of the parabola above, and the equation of the circle with centre $(0, t)$ and radius 1 is

$$x^2 + (y - t)^2 = 1$$

A point (x, y) on both curves satisfies both equations:

$$y = x^2$$

$$x^2 + (y - t)^2 = 1$$



Be sure you understand what this means. For a fixed value of t (a fixed circle) this is a set of two equations in the two unknowns x and y, and the solutions are the intersections belonging to that value of t. For example, for the circle in the diagram at the right, you would expect the equations to have four solutions.

Let's try to solve these equations and see what happens. Note that our solutions will (of course) depend on c. That is we are looking for expressions for x and y in terms of c.

How do we solve two equations in two unknowns? We eliminate one of the variables, giving us one equation in one unknown and we then solve that by isolating the unknown.

So examine the two equations. Which variable should we eliminate?

Well it doesn't matter much. Either way we get the same kind of equation:

Eliminate y: $$x^2 + (x^2 - t)^2 = 1$$

Eliminate x: $$y + (y - t)^2 = 1$$

Let's go with the second equation. It's a lower degree. This is a quadratic equation in y. Put it in standard form:

$$y + (y^2 - 2yt + t^2) = 1$$

$$y^2 - (2t - 1)y + (t^2 - 1) = 0$$

Use the quadratic formula:

$$y = \frac{(2t-1) \pm \sqrt{(2t-1)^2 - 4(t^2-1)}}{2}$$

$$= \frac{(2t-1) \pm \sqrt{4t^2 - 4t + 1 - 4t^2 + 4}}{2}$$

$$y = \frac{(2t-1) \pm \sqrt{5-4t}}{2}$$

Okay. How many solutions do we have? The answer depends on the sign of the discriminant $D = 5 - 4t$.

- $D < 0$ no solutions
- $D = 0$ one solution
- $D > 0$ two solutions

solutions in y

condition		#solns
$D < 0$	$t > 5/4$	0
$D = 0$	$t = 5/4$	1
$D > 0$	$t < 5/4$	2

*Finding t**

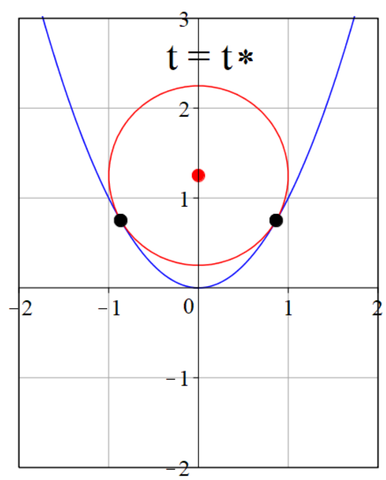
Our critical value t^* is the case of one solution (two intersections with the same y-value).

$$D = 5 - 4t = 0 \qquad \Rightarrow \qquad t^* = 5/4$$

This is the centre of the circle of tangency, and from the diagram that looks about right. The two values of x are symmetrically placed across the y-axis. Indeed since $y = x^2$:

$$x = \pm\sqrt{y}$$

and the two x-values have opposite sign.



To find the intersection points put $t = 5/4$ into the formula for y:

$$y = \frac{(2t-1) \pm \sqrt{5-4t}}{2}$$

$$= \frac{2t-1}{2} = t - \frac{1}{2}$$

$$= \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$$

and then the x-values are

$$x = \pm\sqrt{y} = \pm\sqrt{3/4} = \pm\sqrt{3}/2$$

The two intersections are

$$\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right) \quad \text{and} \quad \left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$$

The student who wants to construct the animation will need to use the general formulae:

$$y = \frac{(2t-1) \pm \sqrt{5-4t}}{2}$$

and

$$x = \pm\sqrt{y}$$