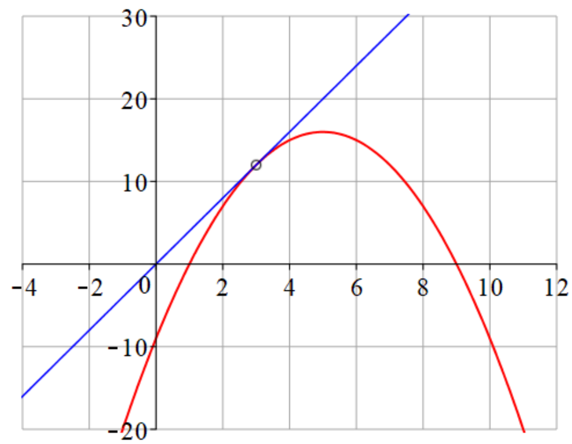
Parabolas and rotating line

At the right I have drawn the graph of the parabola

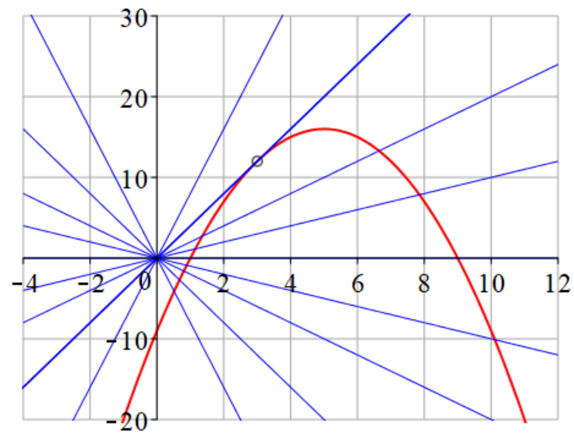
$$y = -(x-1)(x-9)$$

along with a line passing through the origin and tangent to the parabola. Our job is to find the slope of that line.



This might be an interesting calculus problem, but our objective here is to solve it with the tools of algebra. There are a few nice mathematical insights that emerge from this approach.

Our strategy is to consider the family of all lines passing through the origin, and examine the number of intersections each such line has with the parabola. Many of these lines intersect the parabola twice, many have no intersection at all. Our tangent line appears to have exactly one intersection. That should distinguish it.



The lines can be identified by their slope *m* so we use that as the parameter of the family. The family of lines is then:

$$y = mx$$

Now let's intersect each member of this family with the parabola. At any point of intersection, the line and the parabola will have the same height, so we find the intersection by equating the two *y*-values:

$$mx = -(x-1)(x-9)$$

$$mx = -(x^2 - 10x + 9)$$

Put everything on the left and collect powers of *x*:

$$mx + (x^2 - 10x + 9) = 0$$

$$x^2 - (10-m)x + 9 = 0$$

Now it is important to keep our objective firmly in mind. For any particular line (a particular value of *m*) we want the number of intersections of the line with the parabola. These will correspond to the values of *x* that solve the equation for that particular value of *m*.

The quadratic polynomial does not readily factor so we use the quadratic formula:

$$x = \frac{10 - m \pm \sqrt{(10-m)^2 - 36}}{2}$$

Note that as expected, these *x*-solutions depend on *m*. Then here's the question: how many values of *x* does this formula give?

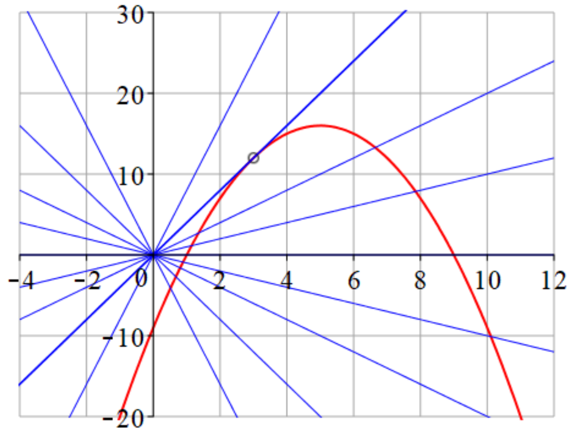
For each fixed value of m, how many values of x does this formula give?

$$x = \frac{10 - m \pm \sqrt{(10-m)^2 - 36}}{2}$$

The key lies in the sign of the discriminant D—the expression under the square-root sign.

$$D = (10-m)^2 - 36$$

- If $D > 0$ we get two solutions for x
- If $D = 0$ we get only one solution for x
- If $D < 0$ we get no solutions for x



Our tangent line, has only one intersection, so it belongs to the case $D = 0$. That can be written

$$(10-m)^2 = 36$$

We want the value of m. Taking square roots

$$10 - m = \pm\sqrt{36} = \pm 6$$

Giving us the two values:

$$m = 10 - 6 = 4$$

$$m = 10 + 6 = 16$$

Which of these is our tangent line? From the graph, the answer is clearly $m = 4$. That's the slope of the tangent line.

What is the x-coordinate of the intersection point? It is

$$x = \frac{10 - m \pm \sqrt{(10-m)^2 - 36}}{2}$$

$$= \frac{10-m}{2}$$

(since $D = 0$)

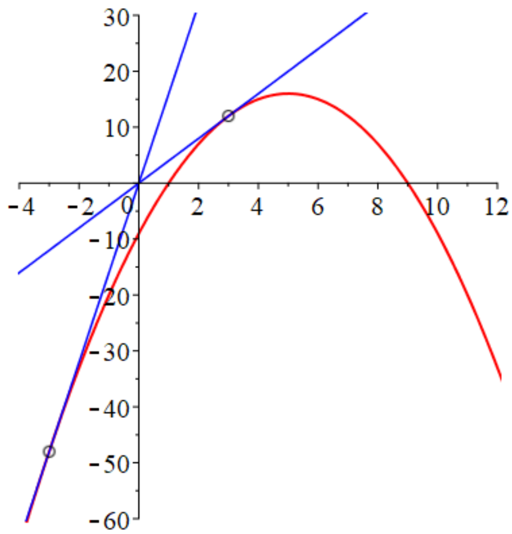
$$= \frac{10-4}{2} = \frac{6}{2} = 3$$

That gives us $x = 3$ and that agrees with the diagram.

But what are we to make of the second value we got for the slope? That was $m = 16$. That's quite a steep slope. Does it give us a tangent to the parabola?

That's a good question to put to the students--can you visualize that second intersection? Draw a rough graph that displays the two tangents.

The other value of *m* does indeed give us a tangent. It has slope *m* = 16 and it intersects the parabola at a negative value of *x*. In fact the *x* value is:



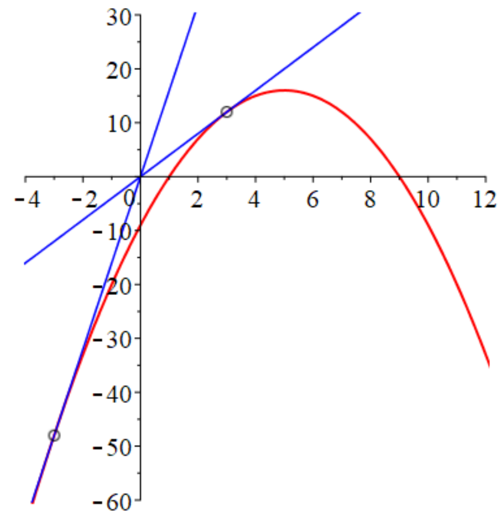
$$x = \frac{10 - m \pm \sqrt{(10-m)^2 - 36}}{2}$$
$$= \frac{10-m}{2}$$
$$= \frac{10-16}{2} = \frac{-6}{2} = -3$$

Hey. That's interesting—or is it? Have you noticed?

Look at the two *x*-values of our points of tangency. They are opposite in sign. They are symmetric about the *y*-axis. Is that a coincidence? Will that always happen?

The answer is yes. And it's a great problem for a good student. It's elementary but it requires careful and inventive thinking. First of all you have to formulate the general result. Here's one version of it.

Take any parabola for which the origin is *outside* the parabola. Then there will be two lines passing through the origin that are tangent to the parabola. And the *x*-values of the two points of tangency will be opposite in sign.



Proposition. Take any parabola for which the origin is *outside* the parabola. Then there will be two lines passing through the origin that are tangent to the parabola. And the *x*-values of the two points of tangency will be opposite in sign.

We work with the general parabola

$$y = ax^2 + bx + c$$

Then the family of lines is

$$y = mx$$

and the intersection equation is

$$ax^2 + (b - m)x + c = 0$$

$$x = \frac{b - m \pm \sqrt{(b - m)^2 - 4ac}}{2a}$$

The condition for exactly one intersection of the line with the parabola is that $D = 0$:

$$(b - m)^2 - 4ac = 0$$

$$(b - m)^2 = 4ac$$

Now to take square roots we must have $ac > 0$. It's good exercise to show that the origin will be "outside" the parabola exactly when $ac > 0$. Given that:

$$b - m = \pm\sqrt{4ac}$$

Now when this holds, $D = 0$ and

$$x = \frac{b - m \pm \sqrt{D}}{2a} = \frac{b - m}{2a} = \frac{\pm\sqrt{4ac}}{2a}$$

And the x-s belonging to the two tangents are opposite in sign.

Finally here is a more general version of the result.

Take any parabola. Take any point P "outside" the parabola. Draw the two tangents from P to the parabola and let them intersect at Q and R. Then the *x*-coordinate of P will be halfway between the *x*-coordinates of Q and R.