3. Recursive thinking.

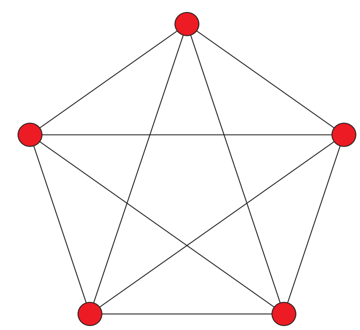
> The trains problem introduced us to the powerful idea of recursive thinking. Here we extend this approach.

(1) The combinatorial coefficients.

The combinatorial coefficient "*n* choose *r*," is written $\binom{n}{r}$ or on your calculator it might be written $_{n}C_{r}$ or $C_{n,r}$. By definition, $\binom{n}{r}$ is the number of ways of choosing *r* objects from a set of *n* objects. Equivalently it is the number of subsets of size *r* in a set of size *n.*

Example 1. Show how the pentagon at the right provides a simple argument that there are 10 ways to select 2 objects from a set of 5:

$$\binom{5}{2} = 10$$



Solution. In the diagram, take the red vertices to be the 5 objects and the 10 connecting lines to be the different ways to choose 2 of these.

Example 2. Suppose the polygon above had 8 sides––an octagon! Would that give you a formula for $\binom{8}{2}$?

Solution. As before we would have an edge between every pair of vertices. Then how many edges would there be? Well each vertex is connected to all 7 remaining vertices so that's 7 edges for each vertex. That seems to be 8×7 edges but each edge is counted twice. Thus the answer is

$$\binom{8}{2} = \frac{8 \times 7}{2} = 28$$

Example 3. Do these examples give you a formula for $\binom{n}{2}$?

Yes they do:

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Example 4 Make list of all the subsets of size 3 of the 6 letters

{A, B, C, D, E, F}.

To do this in a way that allows you to be sure you are correct, your listing needs to be organized. There are many ways to organize it and it's of interest to have different groups report on the way they did it.

One standard way to do it is alphabetical—pretend that they are words in the dictionary. See the table at the right.

ABC	BCD
ABD	BCE
ABE	BCF
ABF	BDE
ACD	BDF
ACE	BEF
ACF	CDE
ADE	CDF
ADF	CEF
AEF	DEF

This tells us that

$$\binom{6}{3} = 20.$$

Pascal's triangle

row0									**1**								
row1								**1**		**1**							
row2							**1**		**2**		**1**						
row3						**1**		**3**		**3**		**1**					
row4					**1**		**4**		**6**		**4**		**1**				
row5				**1**		**5**		**10**		**10**		**5**		**1**			
row6			**1**		**6**		**15**		**20**		**15**		**6**		**1**		
row7		**1**		**7**		**21**		**35**		**35**		**21**		**7**		**1**	
row8	**1**		**8**		**28**		**56**		**70**		**56**		**28**		**8**		**1**

If you don't know about Pascal's triangle, it's time you did. Like the Fibonacci numbers, it has a simple structure, and its entries have some fascinating properties. The top row, with a single 1 is the zeroth row and the rows are numbered 1, 2, 3, … down from there. The last row displayed above is the "8^{th}" row. There are 1's on each end of every row and otherwise each entry is the sum of the two entries above it. That description is enough to completely specify the entries of the triangle.

I ask the class what they know about this triangle and someone tells me that they give us the coefficients in a binomial expansion (known simply as the *binomial coefficients*) while someone else offers that the entries are the combinatorial coefficients we have just been talking about. Indeed we are investigating that one right now.

Pascal's triangle gives us the combinatorial coefficients

The r^{th} entry of row n in Pascal's triangle is $\binom{n}{r}$, the number of ways of choosing r objects from n.
[This uses the convention that indexing starts at zero—the top row of the triangle is row 0 and the left-hand entry of every row is the 0^{th} entry.]

Let's check this out with the examples above. From Pascal's Triangle we see that indeed:

$$\binom{5}{2} = 10$$

$$\binom{8}{2} = 28$$

$$\binom{6}{3} = 20.$$

Of course this raises a big question. If the combinatorial coefficients are found in the triangle, that means they must satisfy the sum law. In that case, there should expect there to be a natural reason for the sum law to hold.

Why do the combinatorial coefficients obey the sum rule?

That's the big question. If the correspondence between Pascal's triangle and the combinatorial coefficients $\binom{n}{r}$ is to be believed, then the combinatorial coefficients must satisfy that simple sum rule. And since everything is so *simple*, there must be a simple way of seeing why this should be true.

row0	**1**								
row1	**1**	**1**							
row2	**1**	**2**	**1**						
row3	**1**	**3**	**3**	**1**					
row4	**1**	**4**	**6**	**4**	**1**				
row5	**1**	**5**	**10**	**10**	**5**	**1**			
row6	**1**	**6**	**15**	**20**	**15**	**6**	**1**		
row7	**1**	**7**	**21**	**35**	**35**	**21**	**7**	**1**	
row8	**1**	**8**	**28**	**56**	**70**	**56**	**28**	**8**	**1**

Let's take a particular example. Can we see why it ought to be true that

$$\binom{7}{3} = \binom{6}{2} + \binom{6}{3}?$$

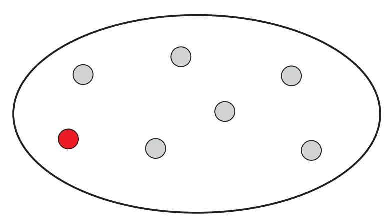
Using the numbers (from the triangle) this is

$$35 = 15 + 20$$

What we need to find is a *combinatorial* argument that the number of ways of choosing 3 things from 7 is the *sum* of the number of ways of choosing 2 things from 6 and the number of ways of choosing 3 things from 6.

How might we show that? On the left we are choosing from a set of 7 and on the right we are choosing from a set of 6. *How can we relate a set of 7 to a set of 6*?

Well here's a way––make one of the 7 different from the others. Colour one of the 7 objects red and colour the remaining 6 grey.



Okay. What are the different ways of choosing a subset of 3 things from that set of 7? Well there are two cases depending on whether the red coin is chosen. If the red coin is chosen we need to choose 2 grey coins and that can be done in $\binom{6}{2}$ ways. If the red coin is not chosen then we have to choose 3 grey coins and that can be done in $\binom{6}{3}$ways. The total number of ways is the sum. And that does it.

Beauty.

The argument is general and shows that:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

This is a simple elegant argument and it's an important one as it gives the students experience at thinking on a structural level. You will find that some students "get it" quickly, in fact they will even come up with the red-grey argument. Others will have trouble. They'll have lost the big picture. They are not sure what we have accomplished that we didn't already know.

(2) **Two Heads in a row**

(a) A coin is tossed n times producing a sequence of heads (H) and tails (T). Let $T(n)$ denote the number of different possible sequences of H and T could result from those n tosses? Find a formula for $T(n)$.

Solution. There are many ways to come to the answer and it's good to have students discuss the various arguments they come up with. One is to start with small n and observe that $T(1) = 2$, $T(2) = 4$, $T(3) = 8$ and $T(4) = 16$ and infer from that that $T(n) = 2^n$.

But can you be certain? One way is to argue that, for example, $T(4)$ has to be twice $T(3)$ because I can take any possible sequence of length 3 and put either an H or a T in front of it, and that way I get all the sequences of length 4. That's a trains-type argument and is quite powerful.

Finally a student might make a slightly more sophisticated argument using probabilistic independence––there are 2 possibilities for the first slot, and for each of these, 2 possibilities for the second, then 2 for the third, etc. giving us the product of n 2's.

(b) Let's call a sequence of tosses "good" if it has two consecutive heads *for the first time* on the last two tosses. Let $G(n)$ be the number of good sequences of length n. For example $G(5) = 3$ as there are 3 good sequences:

HTTHH, THTHH, TTTHH

By similarly enumerating the possibilities, find $G(n)$ for all $n \le 8$ and fill in the table at the right. Can you see a pattern?

# tosses n	*# good* $G(n)$
1	0
2	
3	
4	
5	3
6	
7	

(c) When we fill in the table, we find the Fibonacci numbers. Do they keep going? Can we be certain that $G(8) = 13$?

# tosses n	*# good* $G(n)$
1	0
2	1
3	1
4	2
5	3
6	5
7	8

Solution. What we want is a general argument that

$$G(n) = G(n-1) + G(n-2)$$

And one thinks right away of trying to use a trains-type argument. Let's try to show that $G(8) = G(7) + G(6)$.

Okay. A good sequence of length 8 must either start with a T or with an H. If it starts with a T, the remaining 7 tosses can be any good sequence of length 7, so there are $G(7)$ of those. If it starts with an H, the very next toss will need to be a T (as the first HH must come at the end) and then after that, the next 6 tosses can be any good sequence of length 6. , so there are $G(6)$ of those.

T	7		$G(7)$
H	T	6	$G(6)$
			$G(8)$

Since the total number is $G(8)$, we are finished.

Appendix. The binomial coefficients

Some student know Pascal's triangle as the recipe for expanding a binomial power. For example suppose you want to expand the 6th power of $(a+b)$. You would get:

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

The coefficients we get are the entries in row 6 of the triangle. Our job is to understand why this is the case.

row0									**1**								
row1								**1**		**1**							
row2							**1**		**2**		**1**						
row3						**1**		**3**		**3**		**1**					
row4					**1**		**4**		**6**		**4**		**1**				
row5				**1**		**5**		**10**		**10**		**5**		**1**			
row6			**1**		**6**		**15**		**20**		**15**		**6**		**1**		
row7		**1**		**7**		**21**		**35**		**35**		**21**		**7**		**1**	
row8	**1**		**8**		**28**		**56**		**70**		**56**		**28**		**8**		**1**

Understanding the binomial expansion.

Perhaps the first thing to do is to make sure that everyone understand what $(a+b)^6$ really is:

$$(a+b)^6 = (a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

That is, it's the product of 6 copies of $(a+b)$. Now to get a typical term in the expansion of this product, we take one term out of each of the 6 brackets and multiply them together. And the total number of terms we get is the number of different ways that we can do that. For example, if we decided to take the a out of the first two terms and the last two terms, and take b out of the middle two terms, we'd get the term $aabbaa$ which we'd write as a^4b^2. Of course there are other ways of getting the term a^4b^2, for example we could take a out of the first four terms and b out of the last two. In fact the expansion above tells us there are 15 different ways of getting the term a^4b^2. Can we convince ourselves of that? This is a good question to get a couple of volunteers up to the front to explain the reason to their classmates.

To get 4 a's and 2 b's I need to choose 2 brackets to take the b's from and then I take a from the remaining 4 brackets. In how many ways can I do that? The answer is––the number of ways I can choose 2 things from 6 and that's $\binom{6}{2}$ and that's 15.

The same argument works for the other terms and that gives us the general formula:

$$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6$$

Problems.

1. There are 10 girls and 12 boys in the math club. We want to choose an executive of 4 students.

(a) In how many ways can we do this?
Answer: 7315 ways.

(b) We want the executive to have two girls and two boys. How many ways are there now?
Answer: 2970 ways.

(c) Two of the boys in the club are brothers and another two boys each has one sister in the club. We want the executive to have no sibling pairs (but we are *not* requiring two girls and two boys). How many ways are there now?
Answer 6748

2. [6 marks] Let s_n be the number of sequences of length *n* made up of 1's and 2's that have no consecutive 2's. We will call such sequences "good." For example, of the following sequences of length 9, the two on the left are good and the two on the right are not good.

121111212	122111211
212121212	222211111

n	s_n
1	
2	
3	
4	

(a) By direct enumeration find the first 4 values of s_n and fill out the table at the right.

(b)Making use of the pattern that you find in the table, provide a conjecture for a recursive formula that might generate the s_n (that is, find a formula for s_n in terms of some of the preceding s_k).

(c) Provide a careful argument for your conjecture. Note that you are not asked to *solve* the recursion, but to make an argument, based on the property of the sequences, that your recursive formula must hold. If you don't want to work with a general *n*, you can do what we often do in class and work with particular numbers.

Solution

(a) By direct enumeration we fill in the table

n	s_n
1	2
2	3
3	5
4	8

(b) We are struck by the fact that we have the Fibonacci property—each term being the sum of the two previous terms:

$$s_n = s_{n-1} + s_{n-2}$$

or

$$s_7 = s_6 + s_5$$

(c) We make the argument for *n* = 7. Every good sequence of length 7 must either begin with a 1 or with a 2. In the latter case the next member of the sequence must be a 1. So we have two kinds of good sequences of length 7: those that start with 1 and those that start with 21.

1							

2	1						

In the first case the rest of the sequence can be any good sequence of length 6, and in the second case the rest of the sequence can be any good sequence of length 5. That accounts for all the good sequences of length 7 sequences so that

$$s_7 = s_6 + s_5.$$

We use the same kind of thinking we used for trains. If it's true that

$$s_7 = s_6 + s_5$$

That suggests that there might be a natural way to partition the set of all sequences of length 7 into 2 kinds, with s_6 of one kind and s_5 of the other

The next two problems seem to need some creative inspiration. Students who are into problem-solving will love them.

n	t_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

3. Below is displayed a left-justified copy of Pascal's triangle. Observe that the train numbers are the sums of the diagonals (these are in 1-1 correspondence with the left-hand 1's). For example, starting at the 1 in bottom left corner, we get:

$$1 + 7 + 15 + 10 + 1 = 34$$

[Put a ruler through the diagonal line of numbers and see that it passes through 34.] Using combinatorial notation, this can be written:

$$\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4} = t_9$$

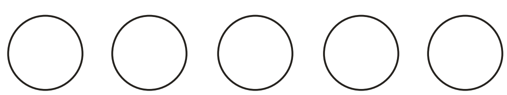
Formulate this in general and find a "train-theoretic" proof.

1	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Solution. Somehow there must be an elegant way to partition the 34 trains of length 8 into 5 sets, each having the different numbers of elements found in the 5 summands. For example, the number in the third set must be easily seen to be equal to the number of ways of choosing two objects from 6.

The argument is surprisingly simple. Condition on the number of 2-cars in the train. For example, to get the fourth term, suppose there are three 2-cars. Then there will be two 1-cars and thus five cars in all. To make the train we have to decide which of those five cars will be the three 2-cars, and to do that we have to choose three slots out of five:



Here is one such choice.

There are ${}_5C_3$ choices in all and that’s the fourth term in the sum. The terms in order correspond to having zero 2-cars, one 2-car, two 2-cars, three 2-cars and four 2-cars.

n	t_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

4. Below is displayed another left-justified copy of Pascal's triangle. In this case the train numbers are ranged both along the top and also (every second number) down the right. Consider the row that starts 1, 5. Take each of these numbers and multiply it by the train number in the top row, and then add all these products. We get the train number at the right. The equation for this case is:

$$(1 \times 1) + (1 \times 5) + (2 \times 10) + (3 \times 10) + (5 \times 5) + (8 \times 1) = 89$$

Using combinatorial notation, this can be written:

$$1\binom{5}{0} + 1\binom{5}{1} + 2\binom{5}{2} + 3\binom{5}{3} + 5\binom{5}{4} + 8\binom{5}{5} = t_{10}$$

Formulate this in general and find a "train-theoretic" proof.

1	1	2	3	5	8	13	21	34

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

1
2
5
13
34
89
233
610
1597