

Red ball, blue ball

At time $t = 0$ a red ball is projected from the origin and travels along the line $y = mx$ at constant speed 2. At the same moment a blue ball is dropped from the point $(4, 4)$ and travels down the line $x = 4$ at constant speed 1. What value of m will produce a direct collision of the two balls?

There are different ways students might approach this, but a good observation is that the red path (to where the paths cross) must be twice as long as the blue path.

Given that, we need to find the length of each path in terms of the slope m of the line. Using the triangle in the diagram at the right, Pythagoras gives us the length of the red path as

$$\sqrt{4^2 + (4m)^2} = \sqrt{16 + 16m^2} = 4\sqrt{1 + m^2}$$

The length d of the blue path can be found from the equation

$$d + 4m = 4$$

$$d = 4 - 4m$$

Setting the red path to be twice the blue path:

$$4\sqrt{1 + m^2} = 2(4 - 4m)$$

$$\sqrt{1 + m^2} = 2 - 2m$$

Now square both sides

$$1 + m^2 = (2 - 2m)^2 = 4 - 8m + 4m^2$$

$$0 = 3 - 8m + 3m^2$$

$$3m^2 - 8m + 3 = 0$$

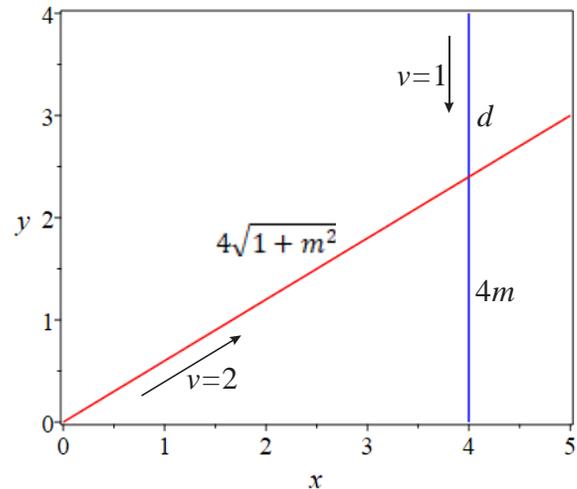
Use the quadratic formula:

$$m = \frac{8 \pm \sqrt{(-8)^2 - 4(3)(3)}}{6}$$

$$= \frac{8 \pm \sqrt{64 - 36}}{6}$$

$$= \frac{8 \pm \sqrt{28}}{6}$$

$$= \frac{4 \pm \sqrt{7}}{3} = \begin{cases} 0.451 \\ 2.215 \end{cases}$$



We have two solutions and these are both drawn in the diagram at the right. We clearly want the smaller slope as the larger slope takes us right out of the diagram.

But what in fact *is* the significance of that other root? Does it have any physical meaning for us?

Yes it does and the diagram at the right displays this by extending the y -axis. It is easy to check that this solves our problem if we simply change the direction of the blue ball. That is, both the lengths marked d are half as long as their corresponding red path.

Does this make sense mathematically? Can we see why the math ought to have given us the two solutions?

That's actually a very good question for the class.

Look back at our equations. The equation

$$4\sqrt{1+m^2} = 2(4-4m)$$

Says that the red path is twice as long as the blue path. And since the left side is positive, so must be the right side and that means that m must be less than 1. That tells us that this equation will have only one solution m .

But when we *square* that equation, we lose the positive sign on both sides, and we allow $(4-4m)$ to be negative. That produces the other solution.

