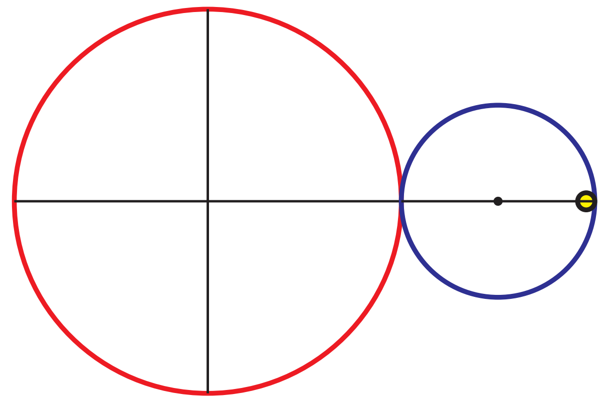
Rolling wheel

This problem features clear structured thinking and the circle interpretation of sin and cos. It gives the students a construction challenge and allows them to see right away, using Desmos, whether they have succeeded. Thus it is reminiscent of Papert's LOGO of the 1970's.

In the diagram at the right, the big circle of radius 2 is fixed and the wheel of radius 1, rotates around it without slipping in a counter-clockwise direction at constant angular speed. The tiny marker disk of radius 1/10 allows us to keep track of the orientation of the wheel. The problem is to use Desmos to construct an animation of the process. The nice thing about this problem is that when you finally get it to work, and see the wheel rolling around the big circle, it's kinda neat.



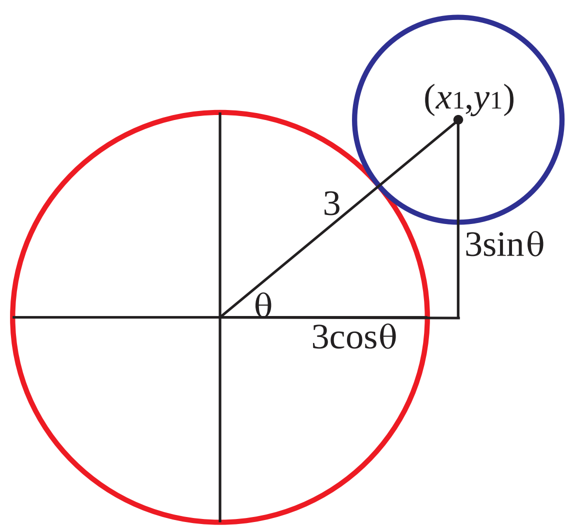
Notice that I ask the students for the animation right away, instead of posing a bunch of intermediate questions. The construction mandate will in fact lead the students to the right questions.

I might say that this process of finding the questions is typically more difficult for the student than the instructor might suppose as the student must first struggle to obtain a rich enough *structural* understanding of the configuration. We mathematicians automatically tackle a problem with such a structural lens, but the student is in fact in the process of developing this powerful way of thinking.

I suggest that they take the origin as the centre of the big circle so that its equation is

$$x^2 + y^2 = 4$$

Taking the angle θ as time, perhaps the next thing to do is to find the coordinates of the centre of the wheel at time θ. That's not so hard as it follows a circle of radius 3 about the origin. Thus:



centre of wheel $\begin{cases} x_1 = 3\cos\theta \\ y_1 = 3\sin\theta \end{cases}$

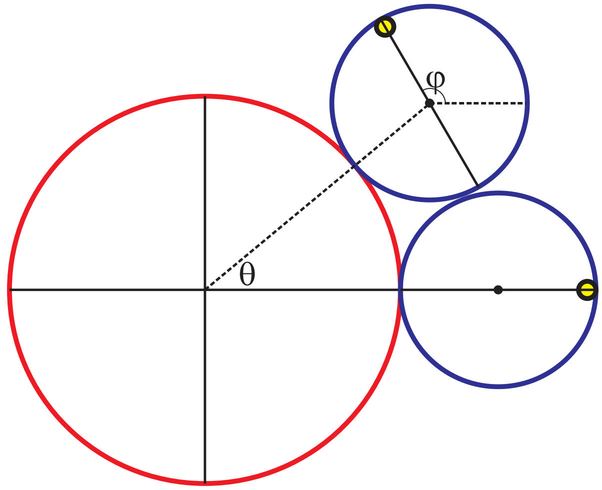
Thus the equation of the wheel is then

$$(x - x_1)^2 + (y - y_1)^2 = 1$$

Note that I decided to introduce the variables x_1 and y_1 to simplify the wheel equation and allow us to see its structure more clearly.

Now we need to draw the marker disk and that requires us to track the orientation of the wheel.

As the wheel travels around the big circle it will rotate and what we need to know is the relationship between the angle θ and the resulting angle of rotation φ of the wheel.



One thing to note is that we expect these to be proportional. That is, if the wheel travels at constant angular speed around the big circle, it will rotate at constant speed as well. To deduce this we are using our intuition. But is it easy to prove? At an OAME talk a few years ago, that question was posed by Ross Issenager and he commented that such "simple" geometric observations are sometimes not so easy to establish rigorously.

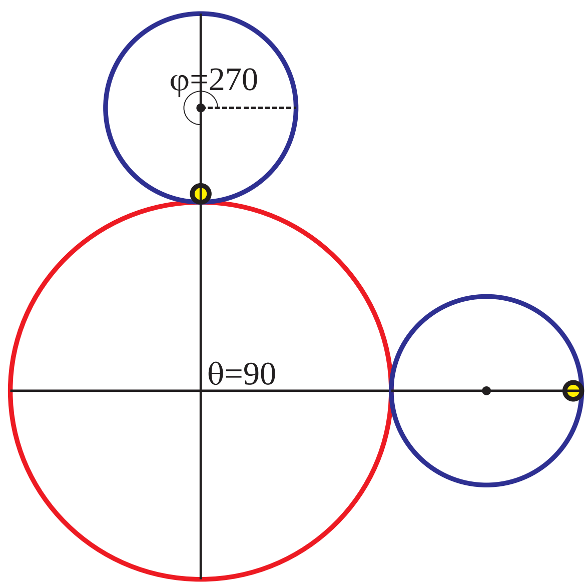
Essentially this proportionality derives from the rotational symmetry of the circle and the wheel. For example, if either of these were an ellipse (or worse, a square) this would not be the case. Anyway, we will not here further pursue the question of proof and will accept the proportionality.

So we have

$$\varphi = k\theta$$

for some constant of proportionality k. We can use any special case to find k. A good choice is $\theta = 90°$.

But here we must think carefully. Many students reason that since the wheel is half the size of the circle, it will turn about its centre twice as fast as it rotates about the centre of the circle. But careful checking will show that it is *the point of contact* that rotates twice as fast as the wheel's displacement around the circle so that when $\theta = 90°$, the tiny circle will be *at the point of contact* with the big circle and that forces φ to be 270°.



Putting this data point in:

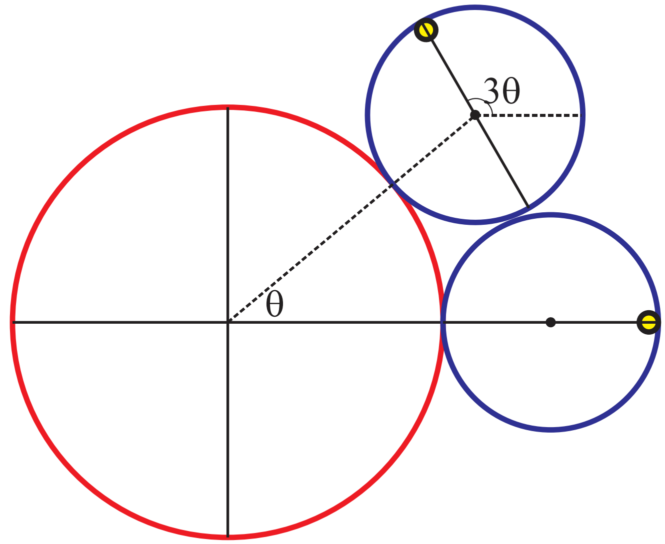
$\varphi = k\theta$: $\quad 270 = 90k$

so that $k = 3$:

$$\varphi = 3\theta.$$

The wheel spins 3 times as fast as it rotates around the circle.

Finally we need to construct the marker disk. The students will need to think a bit about this. In some ways this could be regarded as a vector construction that would fit in the calculus and vectors course. But I'm inclined to think of it as developing preliminary vector ideas.



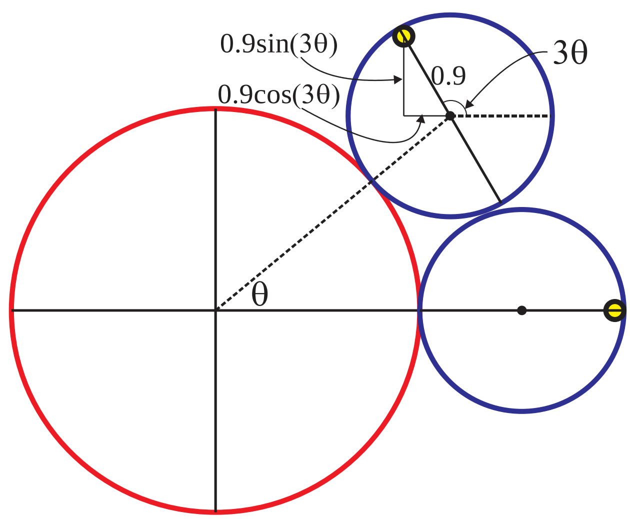
The simplest idea is to start at the centre of the wheel:

centre of wheel $\begin{cases} x_1 = 3\cos\theta \\ y_1 = 3\sin\theta \end{cases}$

and ask where we ought to go from there. Well at time θ we need to head at an angle of 3θ (measured from the positive x-direction).

But how far do we go?––well we go out to the edge of the wheel. But since the marker disk has radius 0.1, we get to its centre after a distance of 0.9. Thus the increments in x and y are

change in x and y $\begin{cases} \Delta x = 0.9\cos 3\theta \\ \Delta y = 0.9\sin 3\theta \end{cases}$



Putting these together we get the centre of the marker disk:

centre of marker disk $\begin{cases} x_2 = x_1 + 0.9\cos 3\theta \\ y_2 = y_1 + 0.9\sin 3\theta \end{cases}$

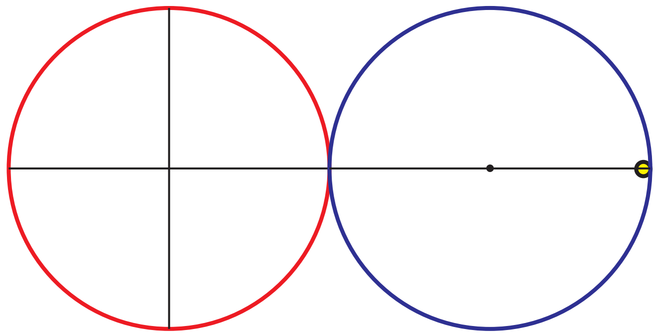
The marker disk has radius 0.1, so it will have equation:

$$(x - x_2)^2 + (y - y_2)^2 = 0.01$$

This is the key equation needed to construct the animation.

Problems

1. In the diagram at the right, the circle on the left of radius 2 is fixed and the wheel on the right, also of radius 2, rotates around it without slipping in a counter-clockwise direction at constant angular speed. The tiny marker disk of radius 1/10 allows us to keep track of the orientation of the wheel. Use Desmos to construct an animation of the process. Provide a fairly detailed explanation so that the reader can see how you constructed the equations. Take θ to be time so that the wheel returns to its starting point in time 360.



Answer

centre of wheel $\begin{cases} x_1 = 4\cos\theta \\ y_1 = 4\sin\theta \end{cases}$

The equation of the wheel:

$$(x - x_1)^2 + (y - y_1)^2 = 4$$

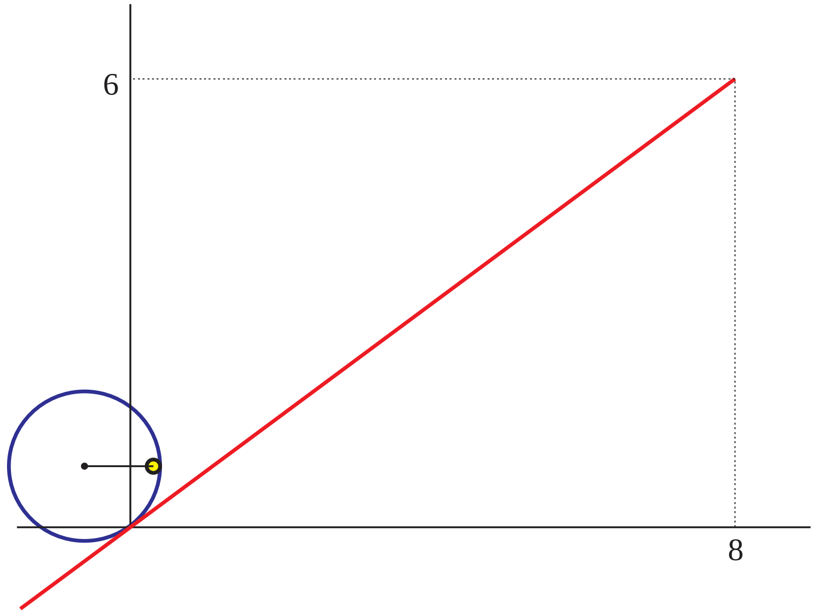
Letting φ be the angle made by the marker circle at time θ, we have $\varphi = 2\theta$. Hence

centre of marker disk $\begin{cases} x_2 = 4\cos\theta + 1.9\cos 2\theta \\ y_2 = 4\sin\theta + 1.9\sin 2\theta \end{cases}$

The marker disk has radius 0.1, so it will have equation:

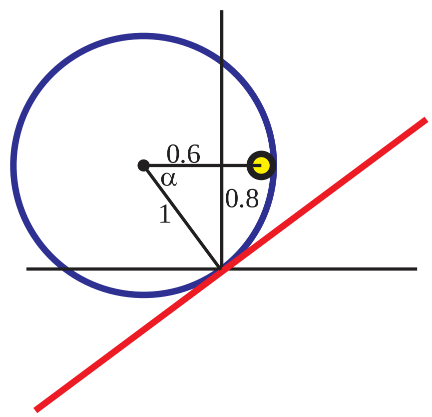
$$(x - x_2)^2 + (y - y_2)^2 = 0.01$$

2. *Wheel along ramp.*
A wheel of radius 1 rolls up a ramp with slope ¾. It begins sitting at the origin and rolls up to the point (8, 6). A small marker circle of radius 0.1 (with centre at distance 0.9 from the centre of the wheel) starts at the same height as the centre ($\theta = 0$) Use Desmos to animate the journey.



To begin we need to choose a suitable time parameter t, and I take t to be the x-coordinate of the point of contact of the wheel with the ramp. To capture the motion of the centre of the wheel we need to calculate its starting position (t=0). The triangle drawn below is a 3-4-5 triangle (similar to the big 6-8-10 triangle) with hypotenuse 1 so its legs are 0.6 and 0.8 as shown. Thus the centre of the wheel starts at the point (–0.6, 0.8) and its trajectory (a, b) is

$$a = t - 0.6 \qquad b = \frac{3}{4}t + 0.8.$$



The dynamic equation of the wheel is then

$$(x-a)^2 + (y-b)^2 = 1$$

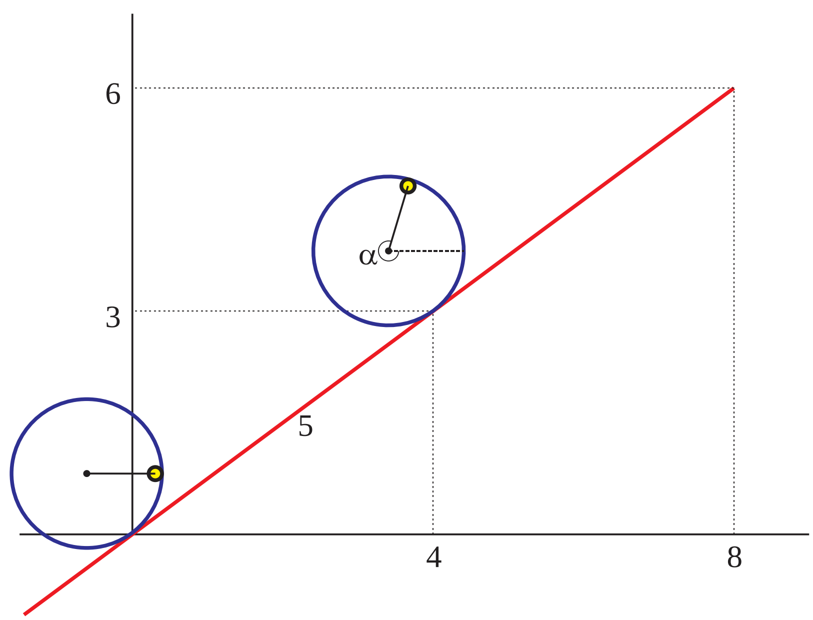
The marker circle. The trajectory of the centre (c, d) of the marker circle needs to incorporate both the translation and the rotation of the wheel. Thus it will have the form:

$$c = a + 0.9\cos(\alpha) \qquad d = b + 0.9\sin(\alpha)$$

where α is the angle the marker-circle ray makes with the horizontal (θ=0) at time t. The dynamic equation of the marker circle is

$$(x-c)^2 + (y-d)^2 = 0.01$$

Finding α. We need to know how much the wheel has rotated at any time t. It's good to have a diagram to look at so consider the wheel at $t = 4$. The angle α through which it has rotated is displayed at the right. How do we see what that angle is? To answer that note that the arc subtended by that angle equals the distance along the ramp that wheel has moved. That's because the wheel does not slip *and the ramp is a straight line.* And since the wheel has radius 1, that arc-length is equal to the angle α (in radians). Finally by Pythagoras, that's 5. In general, that argument gives us $\alpha = -\frac{5}{4}t$.



https://www.desmos.com/calculator/h6x6nj5fpq

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