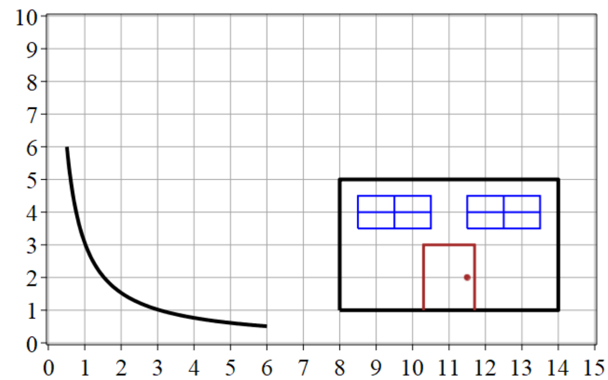
Roofing

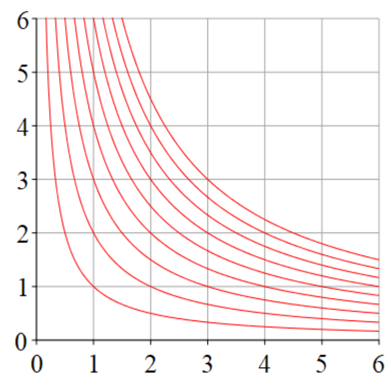
Here is a sad house with no roof. In fact the roof is lying at the side waiting to be installed. Your job is to move it into place.



Here's what you know. The roof graph is a member of the family of curves:

$$xy = k^2.$$

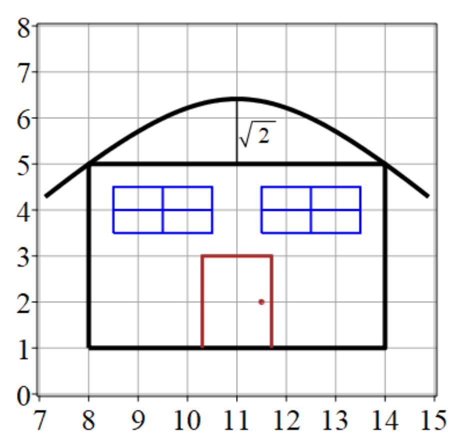
A number of members of this family are plotted at the right. Can you guess what the different k-values are?



[Why did I use k^2 as the parameter instead of k? Well I started off using k but then found that my subsequent calculations involved $\sqrt{k}$. So I switched to k^2.]

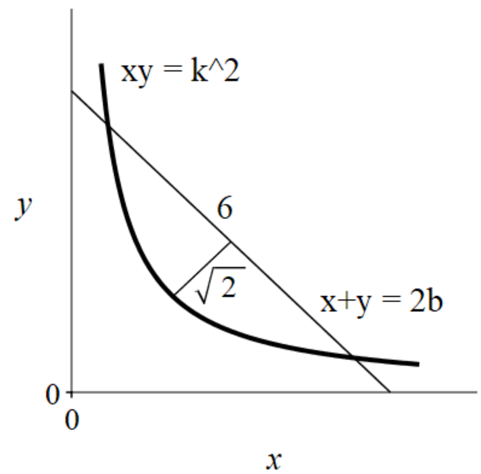
What you need to find is the value of k.

Looking through your files you found something else––a schematic of the installed roof showing the vertical distance between the peak and the flat roof to be $\sqrt{2}$.



That, together with the fact that the house has width 6, should give you all the information you will need.

Find the value of k.



To represent the flat roof on the graph of

$$xy = k^2$$

we need a line of slope −1 (using the symmetry of the graph in the diagonal line $y = x$). Let's take the line to be

$$x + y = 2b.$$

[Why did I use $2b$ as the parameter instead of b? Well just as before, I started off using b but then found that my subsequent calculations involved $b/2$. So I switched to $2b$.]

Okay. We have two unknowns k and b, and two conditions, the distances 6 and $\sqrt{2}$. Hopefully that will allow us to find both parameters.

In terms of the graph, we have two line-segments, of length 6 and $\sqrt{2}$, and to effectively use these we will need the endpoint of those segments.

The length-6 condition.
The segment of length 6 runs between the two intersections of the line and the curve. To find these solve the two equations:

$$xy = k^2 \quad \text{and} \quad x + y = 2b$$

Solve for y and equate the two y-values:

$$\frac{k^2}{x} = 2b - x$$

$$k^2 = 2bx - x^2$$

$$x^2 - 2bx + k^2 = 0$$

We get a quadratic equation in x. Use the quadratic formula:

$$x = \frac{-(-2b) \pm \sqrt{(-2b)^2 - 4k^2}}{2}$$

$$= \frac{2b \pm \sqrt{4b^2 - 4k^2}}{2}$$

$$= \frac{2b \pm 2\sqrt{b^2 - k^2}}{2}$$

$$x = b \pm \sqrt{b^2 - k^2}$$

After all that algebra, what have we got? Well we've got two values of *x* and these are the *x*-coordinates of the two intersection points.

$$x = b \pm \sqrt{b^2 - k^2}.$$

What are the two *y*-values?

Well *they are the same two values*! Both equations are symmetric in *x* and *y*––if you interchange *x* and *y*, the equations don't change. That's a statement of *algebraic* symmetry and of course there also a *geometric* symmetry: if you interchange the *x* and *y* axes, the graphs don't change. If you like, the line and the curve are both symmetric about the diagonal line $x = y$.

Now when *x* is small (at the top end of the curve) *y* is large, and when *x* is large (at the bottom end of the curve) *y* is small. Thus, when *x* gets the minus sign, *y* will have the plus sign, and vice-versa. So the intersection points are

$$x_1 = b - \sqrt{b^2 - k^2} \qquad y_1 = b + \sqrt{b^2 - k^2}$$

$$x_2 = b + \sqrt{b^2 - k^2} \qquad y_2 = b - \sqrt{b^2 - k^2}$$

We are ready to use the distance-6 condition. The square of the distance between the two intersection points must equal $6^2 = 36$:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 36$$

Now,

$$x_2 - x_1 = +2\sqrt{b^2 - k^2} \quad \text{and} \quad y_2 - y_1 = -2\sqrt{b^2 - k^2}$$

We get

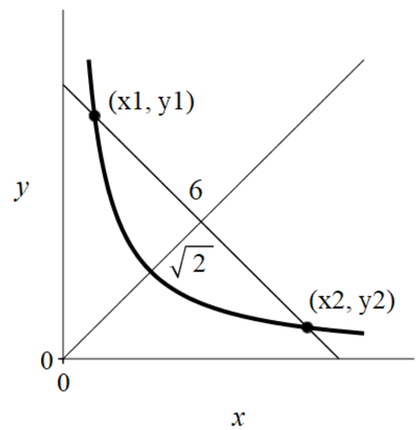
$$\left(2\sqrt{b^2 - k^2}\right)^2 + \left(-2\sqrt{b^2 - k^2}\right)^2 = 36$$

$$8\left(b^2 - k^2\right) = 36$$

$$b^2 - k^2 = \frac{36}{8} = \frac{9}{2}$$

This is our first equation in *b* and *k*.

We have the *x*-coordinates of the two intersection points. Then what are the corresponding *y*–coordinates? That's a great question. Before you start calculating, step back and look the situation over!



The length-$\sqrt{2}$ condition.
The segment of length $\sqrt{2}$ runs between the intersections of the diagonal with the curve and the diagonal with the line. And these are found by setting y = *x* in the two equations:

$$xy = k^2 \qquad \text{and} \qquad x + y = 2b.$$

We get

$$x^2 = k^2 \qquad \text{and} \qquad 2x = 2b$$

and that gives us

$$x = \pm k \qquad \text{and} \qquad x = b$$

We want the +-sign so, since *y* = *x*, the two points are

$$(k, k) \qquad \text{and} \qquad (b, b).$$

The square of the distance between them is 2:

$$(b - k)^2 + (b - k)^2 = 2$$

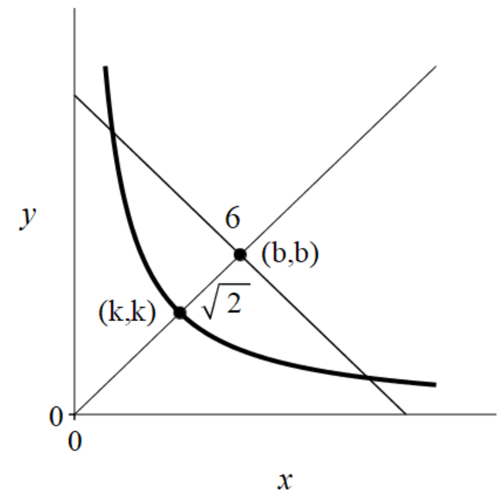
$$(b - k)^2 = 1$$

$$b - k = \pm 1$$

Of course we want the + sign since *b* is clearly bigger than *k*:

$$b = k + 1$$

This is our second equation in *b* and *k*.



Solving the two b-k equations.
We have:

$$b^2 - k^2 = \frac{9}{2} \qquad \text{and} \qquad b = k + 1$$

Eliminate *b*:

$$(k + 1)^2 - k^2 = \frac{9}{2}$$

$$k^2 + 2k + 1 - k^2 = \frac{9}{2}$$

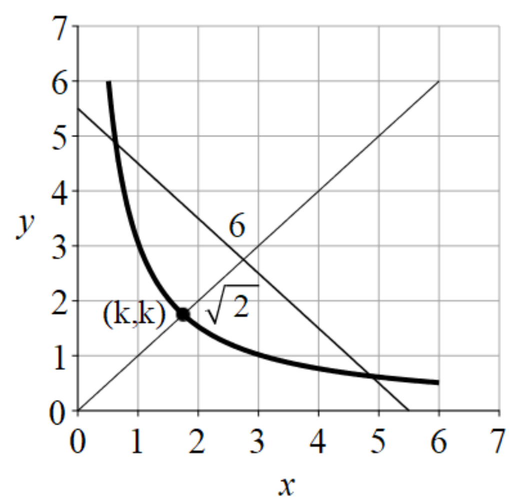
$$2k + 1 = \frac{9}{2}$$

$$2k = \frac{7}{2}$$

$$k = \frac{7}{4} = 1.75 \qquad b = \frac{11}{4} = 2.75$$

The equation of the roof is:

$$xy = k^2 = \frac{49}{16}$$



Below is a Maple animation that moves a copy of the roof into place. The components of the house are all defined ahead of time. The students will be using Python, but the structure of the animation is quite similar.

```
> Roof := proc(t) #lifts the roof into place
  local roof, pt, cable, xshift, yshift :
  if t ≤ 1 then
  #s keeps track of different points on the roof and t measures time [0 < t < 2]. The first half of the journey rotates the roof and translates it
     in a straight line. Note that the rotation is about the origin so the roof first is moved so that its peak is at the origin,
     then it is rotated a proportion t of the whole rotation·(of ang = -0.75 Pi ) and then moved into time t position.
  xshift := k + 6·t : yshift := k + 8·t :
  pt := s→Rotate(s − k, k^2/s − k, ang·t) + (xshift, yshift) :#this rotates and moves each point
  roof := plot([pt(s), s = k^2/6 ..6], colour = black, thickness = 4) :#and this assembles the moving points into a curve with parameter s.
  cable := plot([[xshift, yshift], [xshift, 10]], thickness = 2, color = black) :
  display(roof, cable, house, window1, window2, door, knob, labels = ["", ""], view = [0..15, 0..10], axis[1] = [gridlines = [0, 1, 2, 3, 4, 5,
     6, 7, 8, 9, 10, 11, 12, 13, 14, 15]], axes = boxed, axis[2] = [gridlines = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]], axesfont = [TIMES, ROMAN,
     20]);
  else
  #the second half drops the roof gently onto the house.
  xshift := (k + 6)·(2 − t) + 11(t − 1) : yshift := (k + 8)·(2 − t) + (5 + 2^0.5)·(t − 1) :
  pt := s→Rotate(s − k, k^2/s − k, ang) + (xshift, yshift) :
  roof := plot([pt(s), s = k^2/6 ..6], colour = black, thickness = 4) :
  cable := plot([[xshift, yshift], [xshift, 10]], thickness = 2, color = black) :
  display(roof, cable, house, window1, window2, door, knob, labels = ["", ""], view = [0..15, 0..10], axis[1] = [gridlines = [0, 1, 2, 3, 4, 5,
     6, 7, 8, 9, 10, 11, 12, 13, 14, 15]], axes = boxed, axis[2] = [gridlines = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]], axesfont = [TIMES, ROMAN,
     20]);
  end if
  end proc:

> plots[animate](Roof, [t], t = 0..2, frames = 101);
```