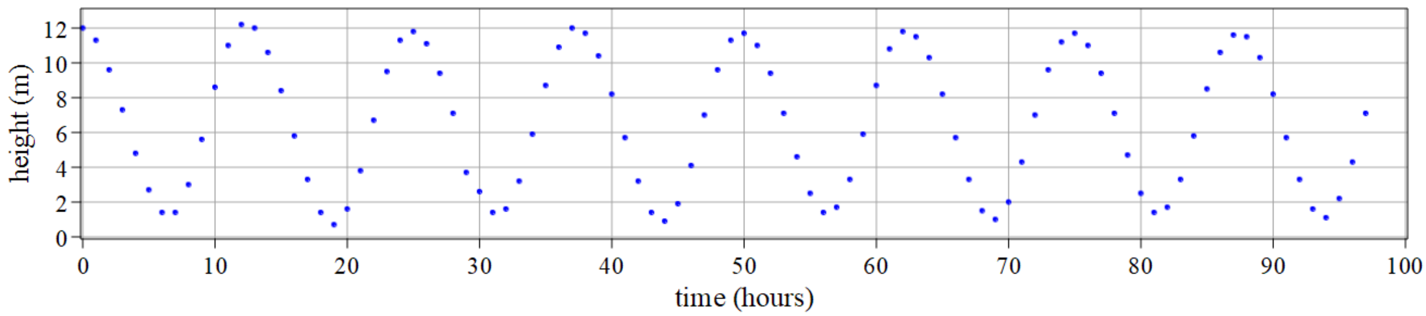
The period of the tides

This is a fascinating journey whose ultimate objective is to understand and even calculate the period *P* of the tides. Find a vertical pier at the edge of the sea, put a scale down the side, and measure the height of the water against time and you'll get a graph something like that below. This is data taken in the Bay of Fundy over a 4-day period. This region is famous for its unusually high tides, a resonance effect caused by the funneling shape of the basin. As you can see from the graph, the amplitude of these tides is almost 6 meters whereas on the ocean, tides are a third or a quarter of that.



The tides are often cited as an example of periodic behavior and in fact they are well modelled with a sinusoidal function. Our first problem is to construct a sinusoidal function which "fits" the above graph. This will give us an equation of the form:

$$h = a\sin(bt + c) + d$$

Now the data, like any data have many sources of error and variation so they aren't *exactly* periodic—as a result we will be looking for reasonable "average" values of the parameters. These variations are due to lots of local things (weather, reading errors, etc.) but there are also some "structural" effects on the periodicity such as the sun. The sun is a large body that most certainly exerts a gravitational force on the earth (if it didn't we'd be in trouble!) and so it affects the tides. But the effect of the sun is much smaller than that of the moon and we will ignore it here.

Just to give you a head start, I will provide reasonable values for two of simpler parameters. I will suppose that the graph oscillates between height h = 1 and y=12. Thus the amplitude is a = 5.5 and the midline is at height h = d = 6.5. I will also observe that the data set starts at t = 0 at height h = 12. Our updated tidal equation has the form

$$h = 5.5\sin(bt + c) + 6.5$$

Your job is to find reasonable values for b and c.

I throw the question out to the class and there is a diverse discussion. A popular choice is to focus on the max and min points of the oscillation. Look for example at the max points. They occur at hours:

0, 12, 25, 37, 50, 62, 75, 87, …

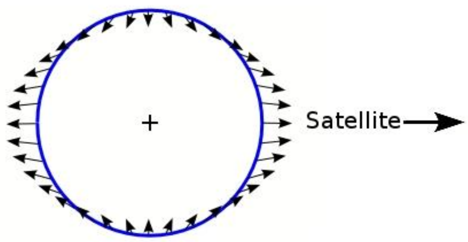
The intervals between them are

12, 13, 12, 13, 12, 13, 12, …

That's a pretty clear pattern and certainly argues for a period of 12.5 hours. Exactly the same differences are obtained from the min points of the graph.

One thing that the students notice in this investigation is that there is recurring pattern of dots in every second max and every second min, and the odd and even patterns are different. That the maxes at 12, 37, 62 and 87 all look alike and the ones at 0, 25, 50 and 75 also all look alike. Where does this interesting periodicity come from? Well it comes from the fact that for one of these sequences the moon is on the same side of the earth as the tide and for the other sequence it is on the other side of the earth.

Both situations cause a high tide. The water on the moon side is closer to the moon than the centre of the earth and has a greater response to the moon's pull. On the other side the water is farther from the moon than the centre of the earth and has a smaller response to the moon's pull, thus the earth in a sense is pulled away from the water. In addition, the near side is closer to the moon than the far side so there is a slight effect for the near-side high tide to be a tiny bit higher than the far-side high tide. You can see that in the graph above as well.



But we are ignoring all those small factors and using a sin function as our model. We have measured the period as 12.5 hours. What value does that give us for the parameter *b*?

Well let's see. We are using degrees here so sin(*t*) has period 360. I can never remember formulae so the first thing I do (to see how things work) is ask what the period is of sin(2*t*). Well, 2*t* will change by 360 when *t* changes by 180. So sin(2*t*) has period 180. That's the same as 360/2. I deduce that

$$\sin(bt) \text{ has period } 360/b$$

We want

$$\frac{360}{b} = 12.5$$

$$b = \frac{360}{12.5} = 28.8$$

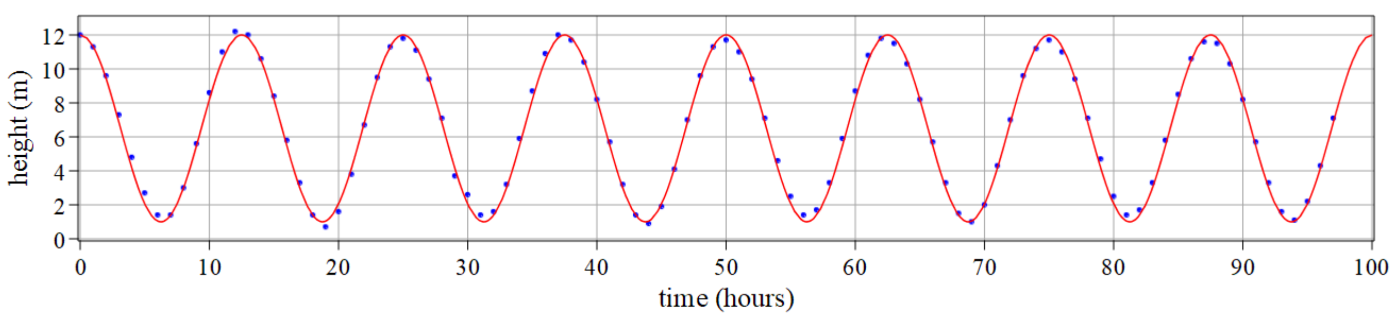
And our formula now reads

$$h = 5.5\sin(28.8t + c) + 6.5$$

Finally we need the phase. We start at 90° so at *t* = 0 the argument of the sin function must be 90. This requires *c* = 90. We get:

$$h = 5.5\sin(28.8t + 90) + 6.5$$

Let's have a look. Below we plot this sine curve on top of the data. It does seem to fit rather well.

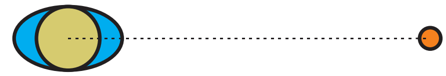


In many ways I find this graph remarkable. We have some data collected from the side of a pier with all sorts of things going on, the sun and the moon, the tilt of the earth, the vagaries of the climate, etc. and nevertheless we get this amazing fit with a simple sin curve. That certainly illustrates the power of mathematics to describe the workings of nature.

But now let's ignore the data and just do the physics and ask what prediction for the tides' period *P* we get from the geometry of the orbit of the moon about the earth.

Some physics and geometry to calculate the period P of the tides

Why are there tides? Because the gravitational pull of the moon on the earth's water stretches it into an elliptical shape and the pull on the earth puts it into the middle of this ellipse. [The water is fluid and the molecules that are closer to the moon get pulled more than the ones that are farther from the moon, but the earth is not fluid and stays circular.] So those on the same side as the moon and on the opposite side will experience a high tide, whereas those in a direction perpendicular to the direction of the moon will experience a low tide. Now as the earth rotates on its axis everyone will experience two high tides every revolution. Thus we need the period of the earth's rotation about its axis—to have some notation I will call that *E*. I ask the students what *E* is and they tell me 24 hours. Okay--given that, this argument predicts two high tides every 24 hours.



This is a greatly abbreviated explanation. Questions arise and there are subtleties to be considered. There's lots about this on the internet.

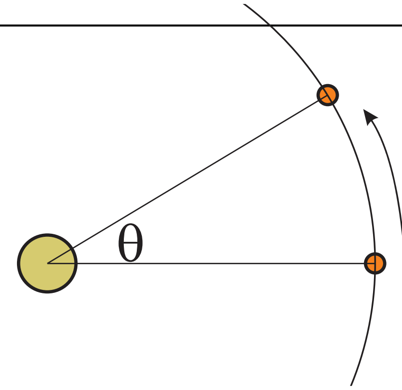
The period of the moon

This argument seems reasonable, but it is missing two important things. The first is that it ignores the fact that the moon revolves about the earth, and to have more notation I will let *M* be the period of that revolution. This turns out to be

M = 29 days 12 hours 44 minutes 2.8 seconds = 708.734 hours

This is a good problem for your students to talk about and play with but it is greatly aided with manipulatives, a few balls of different sizes provide a good start.

The earth-moon system seen if looking down from above the north pole.



Suppose we are sitting "above" the plane of the earth's orbit (with the north pole pointing up). Draw a line between the centres of the earth and the moon and take $t = 0$ to be a moment at which that line intersects the surface of the earth at the zero meridian of longitude (that's the Greenwich meridian) and everyone on it will be experiencing the high point of the tide. Now let the earth rotate through one period (360°) and draw that line again (between the two centres). Will it again pass through the zero meridian?

No it will not, as the moon will have moved in its journey around the earth. Now the moon revolves about the earth in the same direction as the earth rotates about its axis, but much more slowly, so the earth will have to rotate a bit more for the zero meridian to be directly "under" the moon.

But how much more? Let θ be the extra angle through which the earth will have to rotate (and the moon revolve) in order for the zero meridian to be again "under" the moon. The math is simplest if we measure θ in terms of a fraction of a complete revolution. For example if the extra angle were 10 degrees, then θ would be 1/36. How are we going to calculate θ?

Well for the zero meridian to be under the moon again, the earth will need $1+\theta$ rotations and that will take time $E(1+\theta)$. That will have to equal the time the moon will take to go a fraction θ of its complete revolution and that will be $M\theta$.

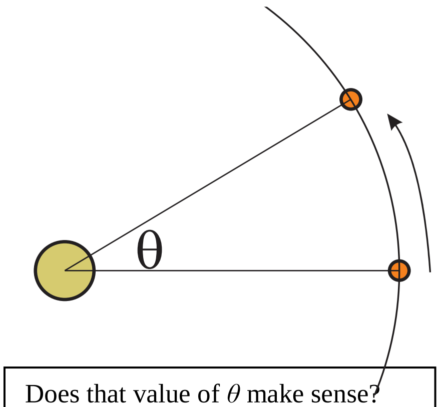
These are the same time so they have to be equal:

$$E(1+\theta) = M\theta$$

$$E + E\theta = M\theta$$

$$E = (M-E)\theta$$

$$\theta = \frac{E}{M-E}$$



Does that value of θ make sense? Let's see—that's close to 1/30th of a revolution. And that's just about the amount the moon moves in one day. That makes sense.

The time required for two high tides is $E(1+\theta)$ and we can now calculate that:

$$E(1+\theta) = E\left(1 + \frac{E}{M-E}\right)$$

$$= E\left(\frac{(M-E)+E}{M-E}\right)$$

$$= E\left(\frac{M}{M-E}\right)$$

$$= 24\left(\frac{708.734}{708.734-24}\right) = 24.8412.$$

The period of the tides will be half of this:

$$P = \frac{1}{2}E\left(\frac{M}{M-E}\right) = \frac{24.8412}{2} = 12.4206 \text{ hours}$$

That's certainly reasonably close to the estimate we got from the Joggins Wharf data. And we don't expect complete agreement as those data do have other factors in them such as the effect of the sun.

I remarked above that there were *two* important things missing from the original naïve argument. We've accounted for the movement of the moon. What else might there be?

The period of the earth—how long is a day?

Well the second factor is rather interesting and that's the period of rotation of the earth about its axis—what we have called *E* in the analysis above.

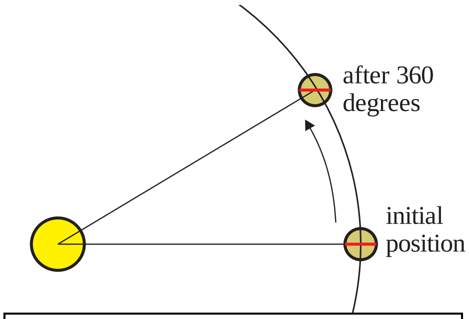
Really? Isn't it just 24 hours. End of story?

Well, not quite. It turns out that there are two kinds of days. Our standard interpretation of the day is what is called the *solar day* because it's determined with reference to the sun. The other kind of day is called the *sidereal day* and it's the time required for the earth to rotate through 360°. It can be defined for any rotating body whether it has a sun or not; in a sense its reference is the rest of the universe.

It turns out that these are different. This can be a good point to turn the issue over to the students. They will have a number of questions but most of these they can get from the internet. Can they figure out what's going on here? Can they make good diagrams to help others understand?

It turns out that we used the wrong value of the earth's period *E* in our calculation above. It's not 24 hours, it's about 4 minutes shorter than that. This is another interesting story and an excellent challenge for student writing and student presentation.

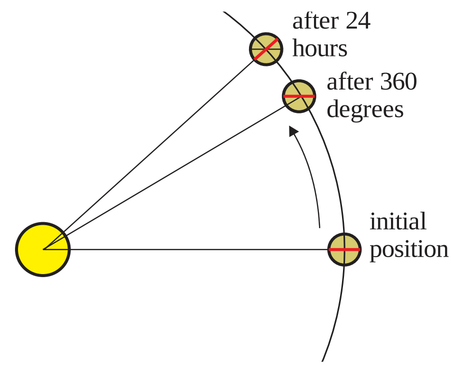
The correction we get from this second part of the story is minor compared with other factors such as the effect of the sun (that we have ignored). But mathematically it's a fascinating excursion.



The sun-earth system seen if looking down from above the north pole. Note that the rotation of the earth about its axis is counter-clockwise as is the earth's revolution around the sun.

Suppose we are sitting "above" the plane of the earth's orbit around the sun (again with the north pole pointing up). Draw a line between the centres of the sun and the earth and take $t = 0$ to be a moment at which that line intersects the surface of the earth at the zero meridian of longitude (coloured red). Now let the earth rotate exactly 360° and draw that line again (between the two centres). Will it again pass through meridian zero? No it will not. You can see that from the diagram (where the amount the earth has revolved around the sun has been greatly exaggerated). For the sun to be directly above the zero meridian, the earth needs to rotate a bit more.

In the second diagram the earth has rotated a bit more so that the line from the sun intersects the 0-meridian. That is what defines the solar day and that's what takes 24 hours. The 360° rotation defines the sidereal day and it's about 4 minutes less than 24 hours. That's what we should have used for E in our moon calculation.

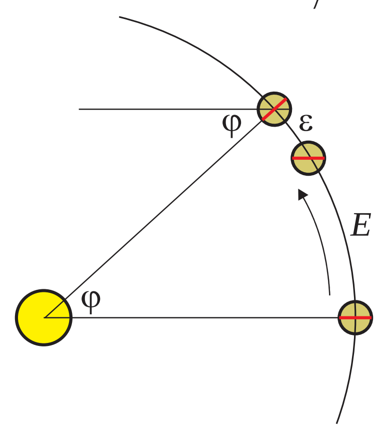


Using geometry to calculate the length E of the sidereal day

This new diagram of the earth's orbit around the sun is the same as above except I have marked the two segments of the solar day with the time in hours they require, E for the first part and ε for the second part. Also the angle through which the earth moves during the solar day—measured as a fraction of a complete 360° revolution— I have called φ.

Since the earth makes 365.25 solar days in its complete revolution ound the sun:

$$\varphi = \frac{1}{365.25} \approx 0.00273785$$



Now we have to think clearly. Our objective is to find E the time for the earth to make exactly one 360° rotation. Now what we know is that $E + \varepsilon$ is the length of of the solar day:

$$E + \varepsilon = 24$$

I guess we need to find ε. Well look what happens to the earth in time ε—it rotates through the angle φ. That comes from a standard theorem in geometry about a line traversing a pair of parallel lines. Now how long does the earth take to rotate through the angle φ? It takes a fraction φ of the time for a complete rotation and that's E. Thus

$$\varepsilon = \varphi E$$

Putting these together:

$$E + \varphi E = 24$$

$$E(1 + \varphi) = 24$$

$$E = \frac{24}{1+\varphi} = 24\frac{1}{1 + 1/365.25} = 24\frac{365.25}{365.25+1} = 24\left(\frac{365.25}{366.25}\right) \approx 23.93447$$

This is the length of the sidereal day. It works out to be

E = 23.93447 hours = 23 hours 56 minutes 4.1 seconds

Given this corrected value of E, we can now correct our calculation of the tidal period:

$$P = \frac{1}{2}E\left(\frac{M}{M-E}\right) = \frac{1}{2}(23.93447)\left(\frac{708.734}{708.734 - 23.93447}\right) = 12.3855 \text{ hours.}$$