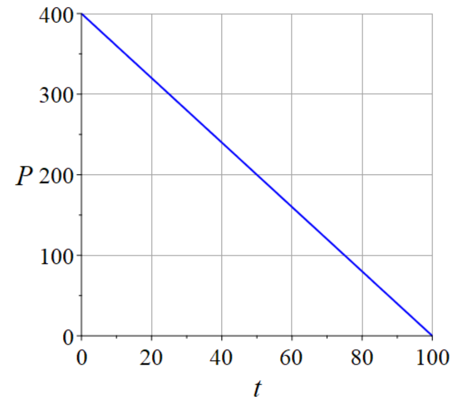
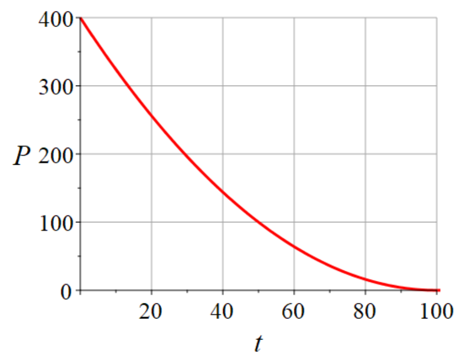
Tire pressure.

You have a hole in your tire. You pump it up to P=400 kilopascals (kPa) and over the next 100 minutes it goes down till the tire is quite flat. Draw what you think the graph of tire pressure P against time t should look like.



I throw this question out to the class and someone (bless his heart) comes up and draws a straight line. That's good because that engenders a debate and after a bit of argument the class settles on the concave-up curve at the right. I ask them to justify the shape and I get the following type of argument.

As air flows out, the pressure P in the tire decreases, but as that decreases, there's less "force" pushing the air out and so the air will flow out more slowly, and as a result the pressure will go down more slowly. Eventually the pressure drops to zero and the graph hits the t-axis.



Okay. Well, that's not too bad. We are in fact going to look more closely at that idea of a "force" pushing the air out.

A model for the tire pressure

Let's get down to basics. The air in the tire consists of a large number of molecules, which are constantly in motion. Now what happens when one of those molecules hits the inside surface of the tire? Well, it bounces off. And in fact that's what causes the pressure in the tire--when you try to push the tire in with your finger it resists and that's because all those molecules are colliding with the inside surface of the tire and pushing it back.

Okay. What happens if there's a hole so that a molecule heading for the inside surface of the tire "hits" the hole instead?

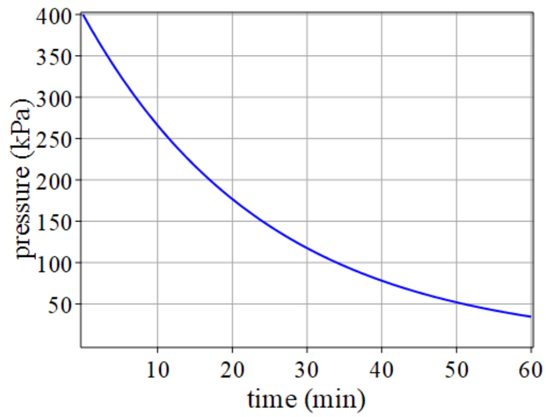
Well it shoots out and escapes. And that's what causes the flow out of the hole. Surprisingly enough, that's all there is to it. Every molecule that flows out is one that was heading for the surface of the tire, minding its own molecular business, and found a hole instead (and tasted freedom!).

Put your finger just above the hole and feel that little jet of air. Are those the molecules that are just escaping by chance? *Yes––that's all there is to it*. Wow.

The students have a tendency to think of the tire as a balloon, with the air being pushed out due to the elastic force of the tube as it contracts. But when the tube is imprisoned inside the tire, it doesn't stretch as it gets filled. If it did, the simple exponential model we are constructing wouldn't apply. It's better to think of the tire as a rigid structure made of hard plastic with air pressure inside.

Constructing the air pressure function—
a thought experiment

Suppose your tire has a small leak. At one point you measure the pressure to be 400 kPa. Suppose over the next minute it drops to 384 kPa ––a loss of 16 kPa. Now suppose you leave it for a while until it's dropped to 200 kPa. Half of what it started with. So here's the question:



How much will it lose in one minute now?

Well, here's a simple argument. The reason that molecules escape from the hole is that in their random motion they happen to run into the hole. When the pressure is 200, there's half as many molecules in the tire as there were when it was 400, so there will be half as many "collisions" with the hole, so the flow rate out should be cut in half. So instead of losing 16 kPa in the next minute, it loses 8.

And so forth. When it has dropped to 100 kPa, it will lose 4 kPa in the next minute. What this argument is really saying is that the amount lost in a minute ought to be always proportional to the amount in the tire at the start of the minute.

A formula for P.

Well, that's a beautifully simple argument, and we can even get an equation out of it. Let's formulate things precisely. We have first said that the pressure *loss* over a one-minute interval is proportional to the pressure at the start of the interval. Alternatively stated:

This is a critical piece of reasoning that we will meet again and again. And many of my first-year students at Queen's did not manage to get hold of it in high school.

every minute the tire pressure drops by a fixed percentage.

What is this percentage for our hypothetical example? Well, when $P = 400$, the loss is 16, and when $P = 200$, the loss is 8, and when $P = 100$, the loss is 4, so the 1-minute loss is always 4% of the starting pressure. Thus:

Exponential growth and decay is about multipliers—there's a multiplier for every time interval and if you know one of these you know everything.

every minute the tire pressure drops by 4%.

Can we construct a formula for $P(t)$ from this? This is a huge question as it is the crux of our work with exponential change. We've seen it before with our study of interest rates and here it is again.

We think in terms of a multiplier.

If the tire pressure drops by 4% every minute then:

$$P(t+1) = P(t) - 0.04\,P(t)$$

Change the decrease into a multiplier:

$$P(t+1) = P(t)(1-0.04) = P(t)(0.96)$$

That tells us that every minute the tire pressure is multiplied by 0.96. And that's something we can easily iterate.

Multipliers

A great way to understand and analyze exponential growth and decay is in terms of multipliers. Every time interval will have a multiplier and there's a simple way of relating the multipliers belonging to time-intervals of different length.

Question: In the example discussed above the 1-minute multiplier was 0.96. What then is the 2-minute multiplier?
Answer: $(0.96)^2$. In two minutes the amount is multiplied by 0.96 in the first minute and again by 0.96 in the second minute, so that's an overall multiplication of

$$(0.96)^2 = 0.9216$$

Question: What is the 10-minute multiplier?
Answer: $(0.96)^{10} \approx 0.6648.$

Question: What is the 30-second multiplier?
Answer: Hmm. That's ½ minute. Well if we let it be *m*, then in 1 minute the amount is multiplied by *m* twice so

$$m^2 = 0.96.$$

Solving:

$$m = (0.96)^{1/2} \approx 0.9798$$

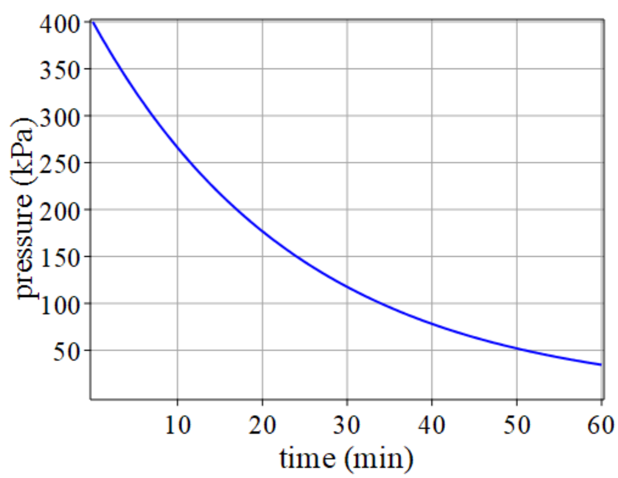
Question: What is the 90-second multiplier?
Answer: Hmm. That's 3/2 minute. So the answer is:

$$(0.96)^{3/2} = 0.9406$$

The general formula should now be clear. After t minutes the pressure will be:

$$P(t) = 400(0.96)^t$$

We have our *P*-function!--it's an exponential decay curve.



Experiment.

The question we are studying is how the pressure *P* in your tire goes down if there's a small leak, and in the last section we developed an exponential decay model. It's time to perform an experiment to check this out.

For that you need a tire. Even more, you need a tire mounted on a wheel (rather than just a tube) because it's essential that the volume of the tire remain the same as the pressure decreases. For most of us, a bike tire is easier to get hold of than a car tire, but a car tire, being a lot larger, definitely gives better data. But if you can't manage to do that, the class can work with the data below.

This data was collected in David Stock's grade 12 class in 1998. We used an old car tire and pumped it up to just under 400 kPa (that's a very hard car tire––they are normally just over 200) and one of the students drilled a hole into the side with a small drill, and it worked perfectly. The pump we used had an in-line gauge so we could leave the pump connected and just monitor the gauge. The gauge was small but it gave us readings accurate to 5 kPa.

We took readings every 5 minutes for an hour. The data are recorded below and plotted at the right. The graph certainly looks reasonable. It has the concave-up shape that our theoretical model predicted.

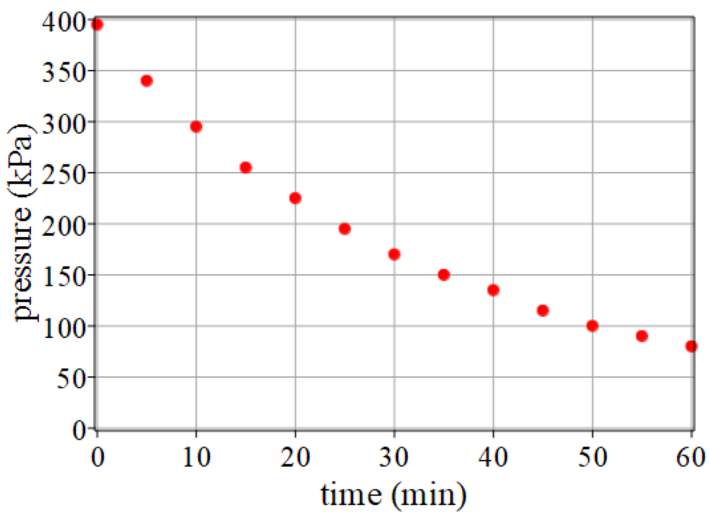
This is a significant experiment, both because it gives good data and because, with a little help, the students can actually come up with a model for the exponential form of the data. So it's well worth making the effort to get hold of a car tire on a rim and keep it around for future years. The same hole will serve for many years.

Who wants the drill?

I held the electric drill up in the air and asked who wanted to drill the hole. I had expected a great rush to the front but there was stillness.

Hasn't everybody always wanted to drill a hole in a car tire? Perhaps they were just awed by the prospect.

time *t* (min)	pressure *P* (kPa)
0	395
5	340
10	295
15	255
20	225
25	195
30	170
35	150
40	135
45	115
50	100
55	90
60	80

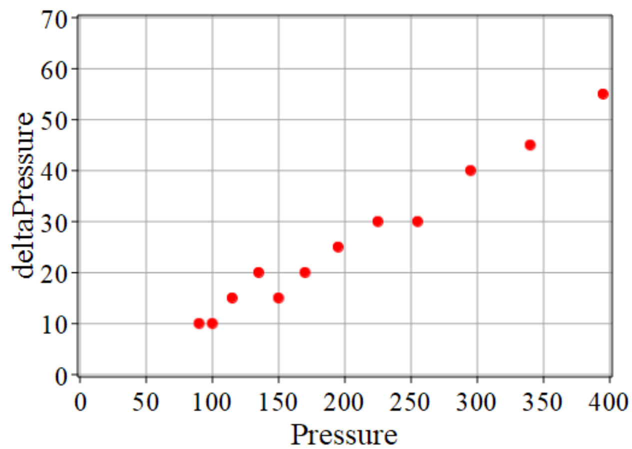


Now what do we want to do with the data? We want to test whether it has the exponential form. So we ask: *Does the pressure decrease at a rate proportional to its size?*

How does the rate of change of P depend on P?
If the pressure did decay exponentially the change in *P* over each time interval would be proportional to *P*. That is, a graph of the 5-minute change Δ*P* against *P* should be a straight line passing through the origin. Let's look at this.

The table on the leftt calculates the change in *P* over each 5-minute time interval against the value of *P* at the start of the interval and these data are plotted at the right. They do seem to lie in a fairly good straight line, though there is considerable scatter when the pressure get's low. Presumably because a 5 kPa error has a larger effect.

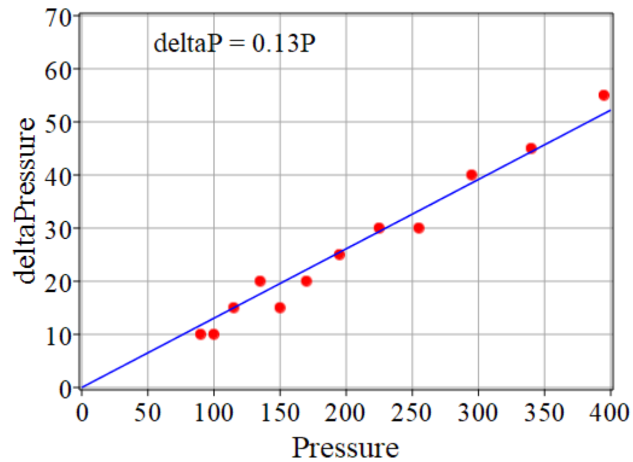
P	Δ*P*
395	55
340	45
295	40
255	30
225	30
195	25
170	20
150	15
135	20
115	15
100	10
90	10
80	



To get a precise average slope for that straight line we calculate a least-squares fit to the data using a line passing through the origin. The line we get is

$$\Delta P = 0.13P$$

and this is plotted below. It looks to me like a good fit and thus it supports our hypothesis that the change in *P* is proportional to *P*.



Now from here we can get a best-fit exponential curve for the data. Here's the argument.

Every 5 minutes *P* is decreased by 13%.

Thus every 5 minutes *P* is multiplied by 0.87.

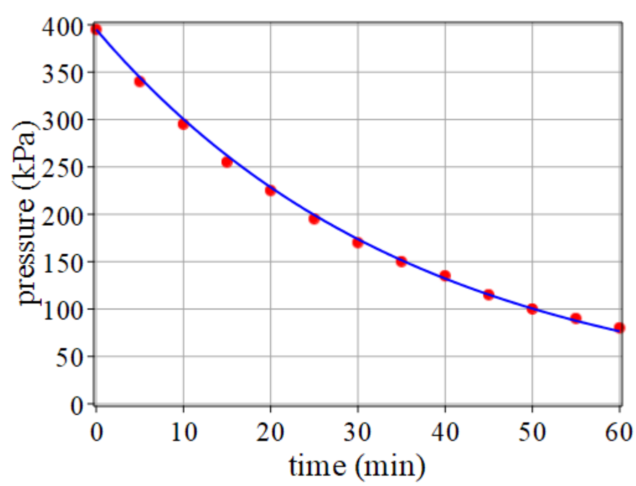
Thus the one-minute multiplier for *P* is:

$$r = (0.87)^{1/5} = 0.973$$

Then our model for the pressure at time *t* is

$$P(t) = P(0)r^t = 395(0.973)^t$$

and this is plotted at the right on top of our amplitude data. It's a great fit.



Problems

1. At the moment (t=0) my tire has pressure P = 300 kPa, but it has a slow leak and loses 5% of its pressure every hour.
(a) What will its pressure be after ten hours (t=10)? [Answer: $P = 300(0.95)^{10} \approx 179.6$]
(b) Find a formula for P after t hours. [Answer: $P = 300(0.95)^t$.]

2. My bike tire has pressure 360 kPa, but after a week this has fallen to 320 kPa.
(a) What will it be after another week?
(b) Find a formula for the pressure P after t weeks.
(c) I can't be bothered to fix the tire and I refuse to ride it when it's below 100 kPa and I don't have a pump and I can only go to the gas station once a week on Saturday mornings, and you have to pay 25 cents for air, so I only fill it up when I have to. Assuming that each pump-up brings it up to 360 kPa, about how long is the interval between pump-ups?

3. **Python** The data on the left was collected from a leaking tire.
(a) Construct a graph of P against t.
(b) Let ΔP be the decrease in pressure over a 5-minute period. Construct a graph of ΔP against P.
(c) Calculate a best-fit line for the data and plot it on top of the data.
(d) Transform the line calculated in (c) into a best-fit exponential equation for P against t and plot its graph along with the P-t data.

t (min)	P (kPa)
0	320
5	290
10	268
15	250
20	235
25	219
30	202
35	190
40	178
45	169
50	157
55	149
60	138
65	130
70	122
75	115

4. I measure my car tire pressure at noon, at 1 PM and at 2 PM. During the first hour it lost 30 kPa and during the second hour it lost 26 kPa. What was the pressure at noon?

t	P
0	A
1	Ar
2	Ar^2

Let A be the starting pressure and r be the 1-hour multiplier. The information we are given is:

$$A - Ar = 30$$
$$Ar - Ar^2 = 26$$

Divide the second equation by the first to get r = 26/30.
Then from the first:

$$A = \frac{30}{1-r} = \frac{30}{1-26/30} = \frac{900}{4} = 225$$

5. I measure my bike tire pressure at noon, at 1 PM at 2 PM and at 3 PM. During the second hour it lost 80% of what it lost during the first hour and during the third hour it lost 50 kPa. What was the pressure at noon?

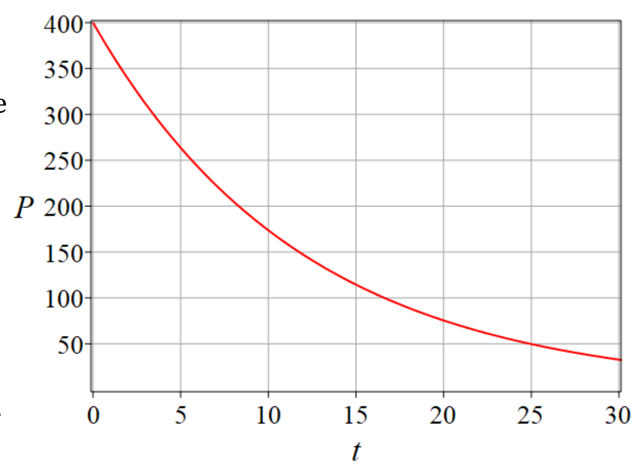
t	P
0	A
1	Ar
2	Ar^2
3	Ar^3

We can use a bit of a shortcut here. The 1 hour loss will be proportional to the pressure at the start of the hour. Since the losses over two successive period have ratio 0.8, the pressures at the start of those periods will also have ratio 0.8. Thus $r = 0.8$.
The third-hour loss is

$$Ar^2 - Ar^3 = 50$$

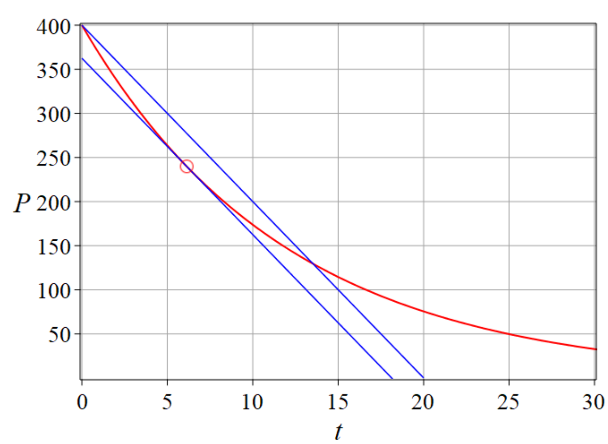
$$A = \frac{50}{r^2 - r^3} = \frac{50}{(0.8)^2 - (0.8)^3} \approx 390.6$$

6. *Small group project.* I have a hole in my tire which makes it lose pressure at the constant percentage rate of 8% every minute. Starting at 400 kPa, the graph at the right shows the pressure trajectory over a 20 minute period. Suppose that, in spite of the hole, I try to keep the tire inflated by pumping air in at a slow constant rate. What I know is that without the hole, and using that same constant rate, the pump will inflate the tire from 0 to 400 kPa in 20 minutes. But if I use the same pump on the tire with the hole, starting at 0 kPa, it never reaches 400 kPa but approaches a constant limiting pressure. Your job is to determine what that limiting pressure might be.



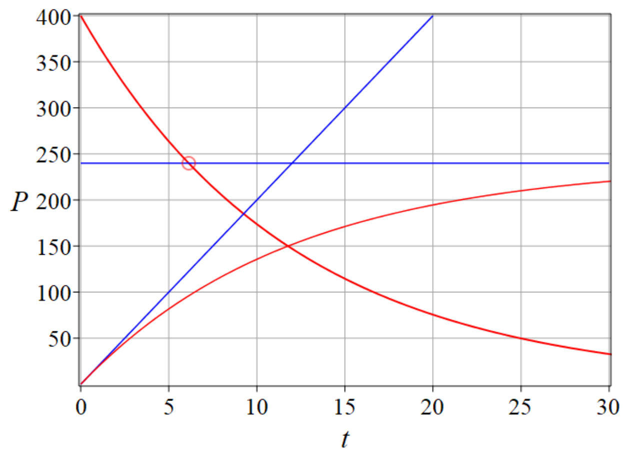
Solution
Work with pencil and ruler on a paper copy of the graph. Determine the limiting pressure and draw a rough graph of the pressure against time for this case (using the pump and starting at 0 kPa). Explaining your conclusions.

At the limiting pressure the pressure doesn't change so there's a balance––the flows of air into and out of the tire will be equal. Now the flow-rate *in* is 20 kPa/min so this will also be the flow-rate out. The flow-rate out is determined by the pressure P in the tire, so that value of P will be at the point on the red graph where the slope of the tangent is –20.



In the graph at the right I have drawn an obvious line of slope –20 and then lowered it just enough so it is tangent to the pressure curve. The resulting pressure seems to be close to 240 so that will be the limiting pressure.

Now let's analyze the trajectory of pumping up the tire with the hole starting at $P = 0$ and approaching the limiting pressure of $P = 240$. At the beginning there is no air in the tire so the pressure should increase at the pump rate of 20. To gauge that, I have drawn a line through the origin of slope 20. If there were no hole in the tire P would continue to increase along this line, but as the pressure increases the flow out of the tire will also increase and the pressure curve will be concave down, and will ultimately approach the constant height of 240.



Equation of the pump-pressure curve.

$$P = 400(0.92)^t$$
$$dP/dt = (\ln 0.92)P$$

Let $k = \ln(0.92)$.
Now with the pump:

$$\frac{dP}{dt} = 20 + kP$$

Remember that k is negative so this is the difference of two positive rates.

$$\frac{dP}{20 + kP} = dt$$
$$\frac{\ln(20 + kP)}{k} = t + c$$

Set $t = 0$. So $P = 0$.

$$\frac{\ln(20)}{k} = c$$
$$\frac{\ln(20 + kP)}{k} = t + \frac{\ln(20)}{k}$$
$$\ln(20 + kP) = \ln(20) + kt$$
$$20 + kP = 20\exp(kt)$$
$$kP = 20[\exp(kt) - 1]$$
$$P = \frac{20}{k}[\exp(kt) - 1]$$
$$P = \frac{20}{\ln(0.92)}[(0.92)^t - 1]$$