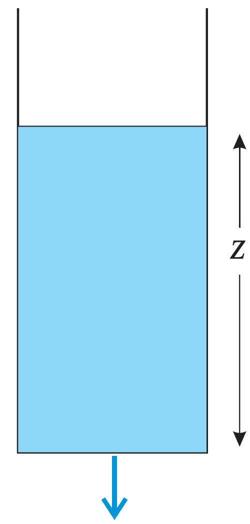
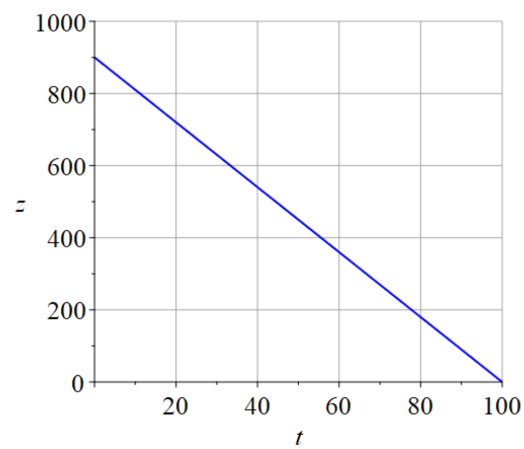
WaterTank



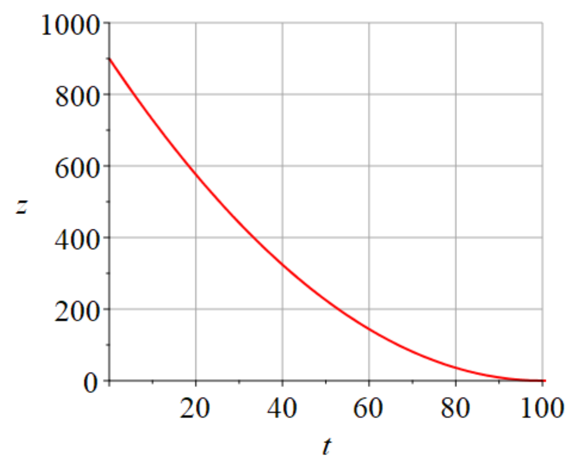
(a) A cylindrical bottle contains 900 ml of water. At *t*=0 (minutes) a hole is punched in the bottom, and water begins to flow out. It takes exactly 100 seconds for the tank to empty. Draw the graph of the water level *z* in the tank against time *t*. Explain the shape of the graph.

I begin, of course, by letting the class watch the action, but without taking measurements. A fine tank can be made from a clear 1 litre pop bottle. I colour the water with food colouring, so it's easier to see from the back. I avoid the non-cylindrical part at the top and the bottom by not filling the bottle right to the top, and by punching the hole in the recessed centre of the bottom so that a residue of water sits permanently in the bottom of the bottle, acting like a fake flat bottom. I drill a hole some 3 to 4 mm (say 1/16th inch) in diameter.



Solution.
I invite a student to come and draw the graph on the board and interestingly enough, what I get is a straight line. I ask the class what they think and there is general cautious nodding. What's good about the graph? It starts at height 900, decreases, and hits the *t*-axis at 100. That seems to fit the given information. So we take a vote. How many agree? I get a forest of hands. Holy cow. That really surprises me.

Yes it did surprise me, but that's really what happened. However this was 1998 and I have a feeling that today I'd get a curve right off the bat. My feeling is that students today have a better feeling for slope and rate of change.



Fortunately there *are* a couple of objectors. *Why is it straight? Shouldn't it be a curve?*

One of these comes up and draws a new picture, and I get something like the graph at the right. Okay. What's better about this graph? Why should it be curved?
Well the flow rate is higher at the beginning than at the end.
Why is that?
More water in the tank—more pressure pushing the water out.
So how does that relate to the geometry of the graph?
Well, when the flow rate is high, z goes down fast, and the graph is steep. And then when the flow rate is less, the graph is flatter.

That's good, and this time it's not hard to get the class to agree that it's right.

How does the decrease in the flow rate over time relate to the curvature of the graph? It's important that they are able to explain that in an intuitive manner.

(b) The form of the equation of z against t depends on the shape of the tank, but for a cylindrical tank, theoretical considerations of how the pressure pushes the water out of the hole lead in fact to the equation of a parabola. Given this, that the z-t graph is a parabola, find its equation.

Solution
The data that we have is that the tank starts with 900 ml of water and that it takes 100 seconds to empty.

Well the equation of a parabola is a polynomial of degree 2, and most of my students start with the algebraic form:

$$z = at^2 + bt + c$$

leaving them three parameters, a, b and c to evaluate. And they can't quite do that because they've only two conditions to work with, that $z(0)=900$ and $z(100)=0$. So they're stuck.

I suggest that they draw the whole parabola (of which only the left half belongs to the water tank problem), and then see what they can say about the equation. When they do that, they realize that the vertex of the parabola must be at t = 100.

Why is that? *Because the flow rate out of the tank determines the slope of the graph and as the tank becomes empty, the flow rate must fall to zero, and therefore the slope at the end-point must be zero.*

Okay. How do we use that information? Well if you have calculus, you can say that the derivative of the function at $t = 0$ must be zero, and that's your third condition. If you don't know calculus, you can say you now know the vertex of the parabola, and given that, there's another algebraic form that's better to use––the completed square form––because the coordinates of the vertex both appear among the three parameters. Since I am a total fan of completing the square, I'll do it that way.

In this case the vertex of the parabola is at (100, 0) so that the equation has the form:

$$z = a(t - 100)^2$$

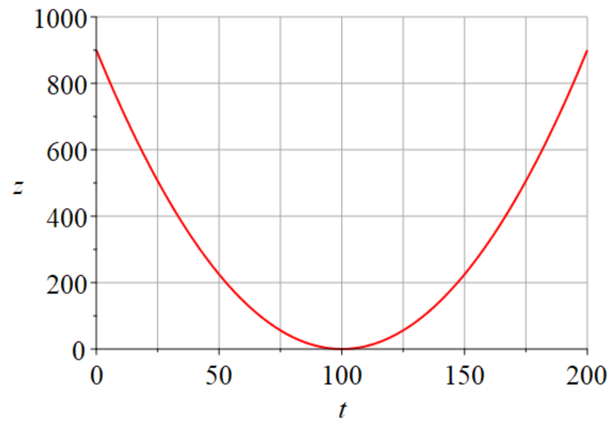
To find a plug in the initial condition t=0, z=900:

$$900 = a(-100)^2 = 10000a$$

and this solves to give a=900/10000 = 0.09. Thus the equation of the parabola is

$$z = 0.09(t - 100)^2$$

This parabola result only applies to cylindrical tanks. *But* the cross-section of the cylinder could be any shape, circular, elliptical, square, triangular etc. As long as the sides are vertical, we call the tank cylindrical, and in that case, the graph turns out to be a parabola.
Actually you don't even need vertical sides. All you really need is that the cross-sectional area of the tank is the same at every height.

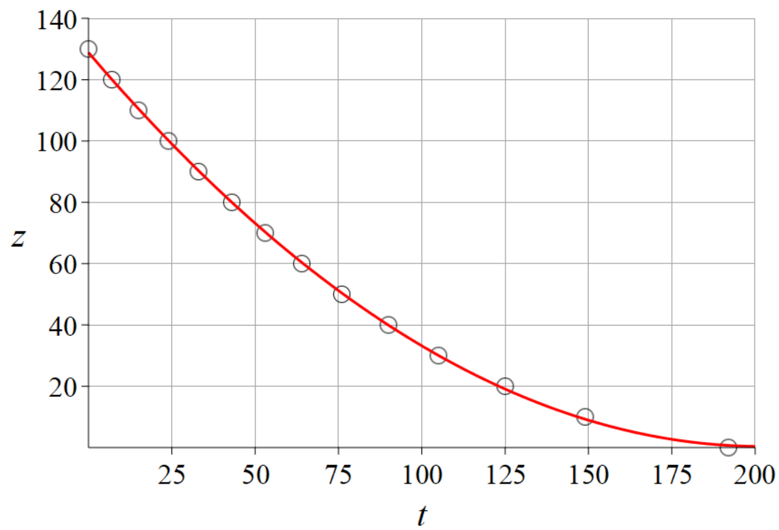


The completed square form

$$z = a(t - b)^2 + c$$

If you are working with a parabola, this is actually a wonderful algebraic form to be given. The parameters a, b and c all describe features of the graph which are geometrically and physically significant. "b" and "c" give you the location of the vertex (t=b and z=c), and "a" tells you how fast the parabola "goes up."

(c) It's time to collect some data and use it to verify the parabola result. The data we collected in the KCVI class is recorded below right. We used a pop bottle with horizontal marks at intervals of 10 mm above the hole and thus we used "height" as a measure of volume. The students recorded the time that the water level arrived at each of the marks. We started at height z = 130 mm and the flow stopped after 192 s. The data is plotted below together with the best-fit parabola.



time t (s)	height z (mm)
0	130
7	120
15	110
24	100
33	90
43	80
53	70
64	60
76	50
90	40
105	30
125	20
149	10
192	0

The best-fit parabola has equation (from Excel):

$$z = 0.0031t^2 - 1.2709t + 128.84$$

And from the diagram it seems to give a pretty good fit.

Let's check the "endpoints." The data give a starting height of z = 130 and the equation gives us $z(0)$ = 128.84. That's a pretty good match. To get the time at which the tank is empty, we set z = 0 and we get:

$$z = \frac{1.2709 \pm \sqrt{1.2709^2 - 4(0.0031)(1.2909)}}{2(0.0031)} = \begin{cases} 183.60 \\ 226.36 \end{cases}$$

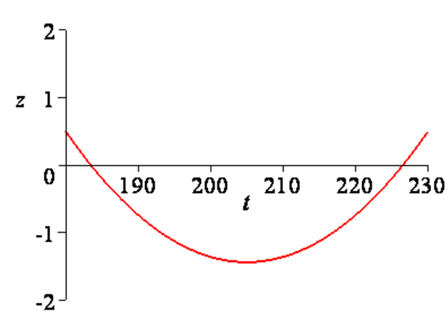
We get two solutions one less than and the other greater than the total time t = 192 given by the data. What are we to make of that?

What that tells us is the best-fit parabola actually dips below the t-axis at the end. A reasonable estimate of the total time might be had at the vertex of the parabola which is at t = 205, halfway between the two roots. That's considerably higher than the value t = 192 given by the data.

The best-fit parabola

Students are much more comfortable with "best-fit" lines and curves than they were when I first worked with problems of this type in the 1990's. No doubt that's partly because they have programs that calculate such things on their calculator or even their cellphone.

One thing they do need is more experience in the analysis of different physical process that would give them an idea of the right functional form.



Thus in a couple of ways, the best-fit parabola does not do so well at the right-hand endpoint. That's at the end of the process when the flow rate is close to zero. How can we understand and possibly even correct this mismatch?

A straight-line analysis

The mismatch we have been talking about occurs when the tank is just empty; indeed it concerns a property of the parabola that we haven't taken advantage of in our mathematical analysis––that the flow rate out of the tank (the slope of the curve) should be zero at the end. Mathematically that means that the vertex of the parabola must sit on the t-axis and that tells us that the parabola has the form:

$$z = a(t - b)^2$$

How do we make technology restrict its attention to parabolas of the above form? That's not so easy to do. The problem is that we only have 2 parameters to work with. But what else has two parameters: *a line!* Can we transform that equation into a line?

I must confess that when working with data I like straight lines. Perhaps that's because when I was a kid, linear fits were the only ones I knew about. But more to the point I can make a pretty good judgement as to whether or not a set of points lies in a straight line, but I'm not so good at recognizing a parabola.

Yes we can––by taking square roots of both sides:

$$z^{1/2} = \pm a^{1/2}(t - b)$$

It's important to remember to include that ± sign, as only one of the two options makes sense. But which is it? Let's think about this. The relevant time interval is that between 0 and b. If we look at the sign of both sides in that interval, we see that everything's positive except (t–b), *which is negative*. So to make both sides positive in that interval, we need to choose the minus sign. Actually a better idea is to turn the (t–b) term around to read (b–t) and then that term would be positive for the relevant t-values:

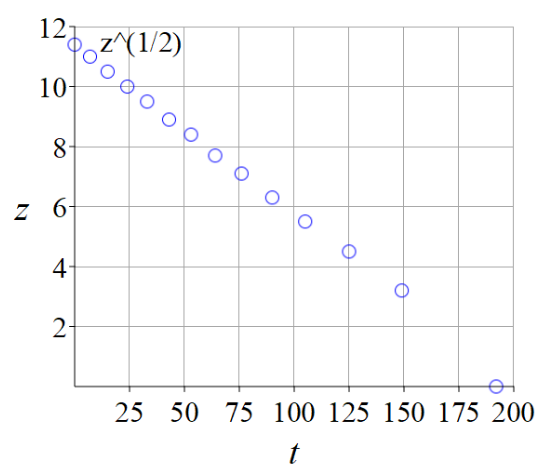
$$z^{1/2} = a^{1/2}(b - t)$$

Now all three terms are positive for the relevant values of t.

t	z	$z^{1/2}$
0	130	11.4
7	120	11.0
15	110	10.5
24	100	10.0
33	90	9.5
43	80	8.9
53	70	8.4
64	60	7.7
76	50	7.1
90	40	6.3
105	30	5.5
125	20	4.5
149	10	3.2
192	0	0

So where have we got to? Do we have a straight line yet or not? Well the right-hand side is linear in t, but the left-hand side is not linear in z. But it *is* linear in the square root of z.

In other words, *if we were to tabulate the values of the square root of z, and plot those against t, we should get a straight line*!



Let's do it!

Wow! There's a real sense of pleasure and excitement in watching that straight line of points emerge. That's actually a beautiful visual "proof" that the original data points lie along a parabola. That's just one of the several reasons I like this approach.

Now we can ask for a *linear* best fit. We get the equation.

$$z^{1/2} = 11.419 - 0.0572t$$

What do you think? Well it's not bad. But something seems to go wrong at the right-hand end––that very last data point is much too low and the two points before it are a bit too high. What going on there?

Well in the annals of data collection, the end points are often the hardest to capture and here we might suspect that it's the right-hand end point that's at fault—if it were higher it would pull the whole line up and everything would be good.

Let's be more careful here. That last point is the $z = 0$ point and it has to be on the bottom line. So it's not that it's too low, it's that it ought to be farther to the right. It ought to have a higher *t*. The flow ought to have stopped at a later time. Does that make any sense?

I throw this question out and get a good discussion. It turns out that this is a graphic illustration of the surface tension effect, the low pressure at the end being insufficient to overcome the viscosity or the liquid. So that in fact a small amount of water remains above the hole at the end.

What do we do to correct this? I get a good discussion from the class about this. That last point is called an "outlier" and the best thing to do is to get rid of it and to use the remaining points to determine the best straight line.

Let's do it. We order up another trend-line using only the first 13 points:

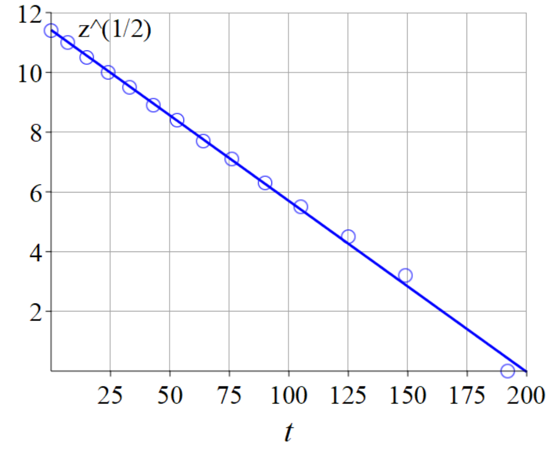
$$z^{1/2} = 11.321 - 0.0552t$$

Notice how much better the fit is to those first 13 points. That's a good sign right there—they really do seem to give us a good parabola [Recall that we are using the linearity of these transformed points to give us measure of how close the original points are to being parabolic.]

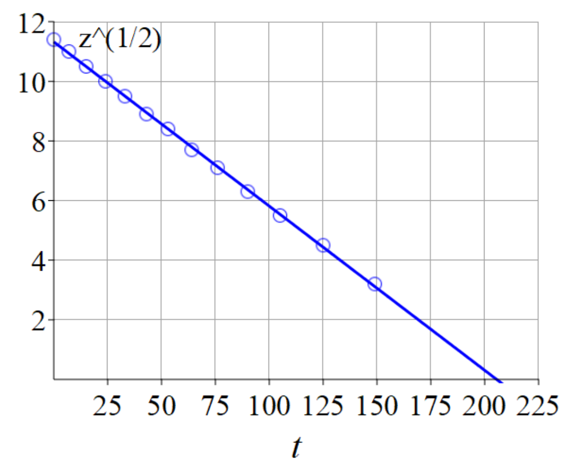
The line gives us a way of determining when the flow "ought" to have ended––the intersection of the line with the *t*-axis. We get the *t*-intercept of

$$t = 11.321/0.0552 = 205.1$$

and that's our prediction of when the tank should have become empty if the surface tension effect hadn't held the water back at the end.



That last data point is definitely off the line, almost flagrantly so. Is it trying to tell us something? *Yes it is!* It turns out that we can find a physical reason to *expect* it to be off the line-- viscosity effects. And thus we have a good justification to kick it out of the data set.



Finally to get the parabola we square the linear equation.

$$z = (11.321 - 0.0552t)^2$$

In fact we want this in the form

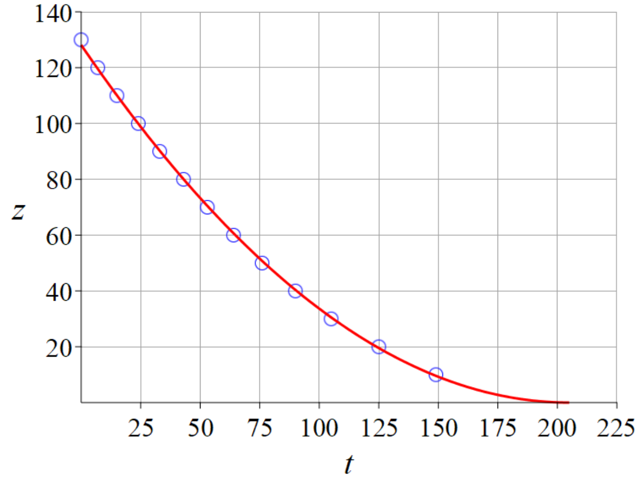
$$z = a(b - t)^2$$

So we factor out the 0.0552:

$$z = \left[0.0552\left(\frac{11.321}{0.0552} - t\right)\right]^2 = 0.0552^2\left(\frac{11.321}{0.0552} - t\right)^2$$

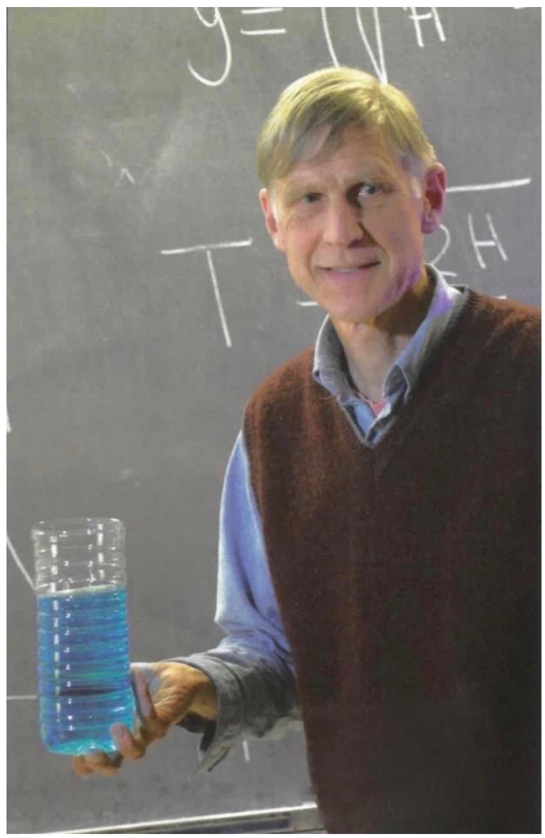
$$z = 0.00305(205.1 - t)^2$$

There's our parabola.



Notes on student activity

I first worked this problem with a real class in 1998 at KCVI in a David Stocks' Grade 12 class with the help of my then PhD student Nathalie Sinclair. We were preparing for the big rewrite of the Ontario curriculum. The account and the data here is pretty well what happened on that occasion. Note that my finger is comfortably in the hole in the middle of the bottom.



In this 2019 rewrite I assume that the students have individual access to technology such as excel and Python and they would be doing most of this themselves paced or guided perhaps by a the action on a laptop on the screen at the front.

Who knows how it will go? Perhaps:

Start with the experiment,
plot the data and fit the parabola,
talk about things,
introduce the form $z = a(t - b)^2$,
take square roots and plot the transformed data,
discuss the plot

Etc.

Problems.

1. A parabola has vertex at x=4, y=2. It intersects the y-axis at y=34. Make a sketch of its graph and find its equation.

2. At t=0 (minutes) a tank holds 1000 L of water. Over the next 5 minutes a total of 500 L is pumped out of the tank at a variable rate so that at t=5 the tank holds only 500 L. The equation of the amount z (litres) in the tank at time t (minutes) has the form

$$x = a(5-t)^3 + b \qquad (0 \le t \le 5)$$

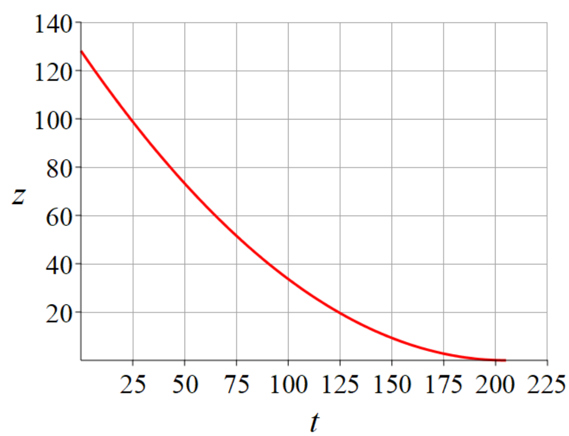
Find the parameters a and b.

3. A tank contains 200 litres of water. Suppose we pump the water out of the tank at the constant rate of 20 L/min. Draw the graph of the amount z in the tank against time t over the 10-minute period until the tank is empty and find the equation of this line. Take the time origin t=0 to be the moment at which the pump begins to operate.

4. A cylindrical tank starts with 800 litres of water. Suppose we pump water out of the tank for 10 minutes at the constant rate of 20 L/min and then for another 10 minutes at the constant rate of 40 L/min.
(a) How much water will there be in the tank at the end of the 20-minute period?
(b) Draw the graph of the amount z in the tank against time t over the 20-minute period.
(c) Find the equation of the line segments.
(d) At what time is the tank half full (i.e. 400 litres)? Illustrate this on a copy of your graph.

5. A cylindrical tank has 900 litres of water. At t=0 (minutes) a hole punched at the base, and water pours out. It takes exactly 60 minutes for the tank to empty. Use the fact that the z-t graph is a parabola to answer the following questions.
(a) Find a formula for z as a function of t.
(b) How much water is in the tank at t=30 minutes?
(c) At what time are there 400 L in the tank?
(d) When is the tank half-full?

6. At the right is a copy of the final parabola we derived in this section. Suppose the water which runs out of the hole is directed into a bucket which starts empty. Draw the graph of the amount of water in the bucket as a function of time t. What is the geometric relationship between the curve that you have drawn and the graph at the right? Be as precise as you can. Illustrate by including both curves in the same diagram.



7. In Example 1 we preferred to work with an equation of the form $z = a(b-t)^{1/2}$ because the parameter *b* has a nice physical interpretation as the time for the tank to empty. But what about the parameter *a*? Well there isn't a nice interpretation for it, but there is another physically significant parameter, and that's the starting height of the water, call it *h*. Here's the question: find a form of the general equation which uses these two nice parameters *b* and *h*, instead of *b* and *a*. [And the *b*-*h* form is actually the "natural" form to work with.]

8. A tank with a hole in the base is empty. Starting at time t=0, water is run into the tank at a constant rate. After a reasonable time has passed, the water level in the tank seems to be constant at depth 100 mm. Draw the graph of the water level in the tank as a function of time for $t \geq 0$.

9. Two identical cylindrical tanks, A and B each contain 800 ml of water. One hole is punched at the base of A and two holes are punched at the base of B, all three holes of the same size. On a single set of axes, draw graphs A and B of the amounts of water in each tank against time. What relationship exists between the slopes of graphs A and B? Illustrate this relationship by referring to specific points on the graphs.

10. I have two cylindrical tanks, A and B, the first with twice the cross sectional area of the second. The tanks have identical holes at the base, and each starts with 800 ml of water. On a single set of axes, draw graphs A and B of the amounts of water in each tank against time. What relationship exists between the slopes of graphs A and B? Illustrate this relationship by referring to specific points on the graphs.

11. The data set at the right is supposed to fit a model of the form

$$z = a(b-t)^{1/2}$$

but there are random errors in the collection and no reliable figure was obtained for the *t*-value when z=0. Transform the data to fit a straight-line model, and use ruler and eye or a trend-line routine to plot the data, obtain the best-fit straight line, and estimate the parameters *a* and *b*. What's your best guess for the time at which z=0?

t	z
0	60
10	57
20	55
30	52
40	49
50	46
60	43
70	40
80	37
90	32
100	28
110	22
120	15
?	0

12. I have some x, y data which are supposed to fit an equation of the form $y = a(x-b)^2$ for some suitable values of the parameters a and b. So I plot $y^{1/2}$ against x and I get a fairly good straight line. The regression equation is $y^{1/2} = 8.32-0.20x$. What estimates of a and b does this line gives us?

13. A large cylindrical tank containing 100 litres of water empties through a hole at the base. The data set at the right shows the time t (minutes) at which the water level is at different heights z (cm). Thus the data are supposed to fit a model of the form $z = a(c-t)^2$. Transform the data to fit a straight-line model, and use ruler and eye or a spread sheet and a regression routine to plot the data, obtain the best-fit straight line, and estimate the parameters a and c.

t	z
0	100
18	90
36	80
54	70
75	60
96	50
121	40
150	30
179	20
215	10
273	0

14. The graph of $z^{0.7}$ against t is a straight line with slope –1.3 and t-intercept t=42. Find an equation for z in terms of t.

15. I have three cylindrical tanks, one with 100 L, one with 200 L, and one with 300 L. Now I punch holes at the base of each of them and let them all start to empty at the same time, the first two empty into pool A and the third empties into pool B. Remarkably enough, all three tanks becomes empty at the same time. If both pool A and B start empty, do they always have the same amount of water?

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