1. In this problem we will use the geometric and algebraic properties of the cross product to deduce the formula for the cross product. In particular, this shows that there is only one possible formula, and shows how it is possible to come up with such a thing.

Let $\vec{e_1} = (1, 0, 0)$, $\vec{e_2} = (0, 1, 0)$, and $\vec{e_3} = (0, 0, 1)$.

(a) Use the right hand rule and the length formula for the cross product to show that $\vec{e_1} \times \vec{e_2} = \vec{e_3}$.

(b) Similarly compute $\vec{e_2} \times \vec{e_3}$ and $\vec{e_1} \times \vec{e_3}$.

(c) Given a vector $\vec{v} = (v_1, v_2, v_3)$ show that $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + v_3\vec{e_3}$.

The cross product satisfies these algebraic rules:

(**antisymmetric**) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ for all $\vec{a}, \vec{b} \in \mathbb{R}^3$.

(**homogeneous**) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$ for all $\vec{a}, \vec{b} \in \mathbb{R}^3$, $c \in \mathbb{R}$.

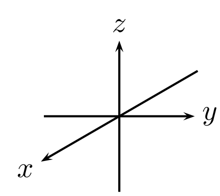
(**distributive**) $(\vec{a}+\vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ and $\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ for all $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.

(d) For any vector $\vec{u} \in \mathbb{R}^3$, show that $\vec{u} \times \vec{u} = \vec{0}$. (Use one of the geometric or algebraic properties.)

(e) Given vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, use the algebriac properties and parts (a)–(d) to deduce what $\vec{v} \times \vec{w}$ must be.

NOTES: (1) In this problem, do *not* use the algebraic formula for the cross product from class — the whole purpose of this problem is to deduce that formula from the properties of the cross product.

(2) There are two ways of setting up the x-, y-, and z-axes when making a three-dimensional drawing. In every problem but this one, it does not matter which way you use. For this problem, please use the axes as shown at right. (If you use the other way of setting up the axes, the cross product obeys a "left-hand rule", rather than a right-hand one.) The labels shown appear in the positive direction of each of the axes.



2. The Cauchy-Schwarz inequality is a very important and useful result about the size of the dot product. It states that for any two vectors $\vec{v}$, $\vec{w} \in \mathbb{R}^n$, the inequality $|\vec{v} \cdot \vec{w}| \leqslant \|v\|\|w\|$ holds. In this problem we will prove the Cauchy-Schwarz inequality in two different ways.

(a) First, deduce the Cauchy-Schwarz inequality from the dot-product theorem.

We now work towards a different proof of the same inequality.

(b) Suppose that a, b, and c are real numbers such that $at^2 + bt + c \geqslant 0$ for all $t \in \mathbb{R}$. Explain why this implies that $b^2 - 4ac \leqslant 0$. (Suggestion: Think about what the inequality is saying about the parabola $y = at^2 + bt + c$.)

(c) Let $\vec{v}$ and $\vec{w}$ be any vectors in $\mathbb{R}^n$. Explain why $(t\vec{v} + \vec{w}) \cdot (t\vec{v} + \vec{w}) \geqslant 0$ for all $t \in \mathbb{R}$.

(d) Use the distributative propery of the dot product to rewrite $(t\vec{v} + \vec{w}) \cdot (t\vec{v} + \vec{w})$ in the form $at^2 + bt + c$. (As part of the problem you will find formulas for a, b, and c in terms of $\vec{v}$ and $\vec{w}$.)

(e) Combine parts (b)–(d) to give another proof of the Cauchy-Schwarz inequality.

(f) Finally, suppose that x_1, x_2, x_3, and y_1, y_2, and y_3 are any real numbers. Show that the inequality

$$|x_1y_1 + x_2y_2 + x_3y_3| \leqslant \left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)\left(\sqrt{y_1^2 + y_2^2 + y_3^2}\right)$$

is always true.