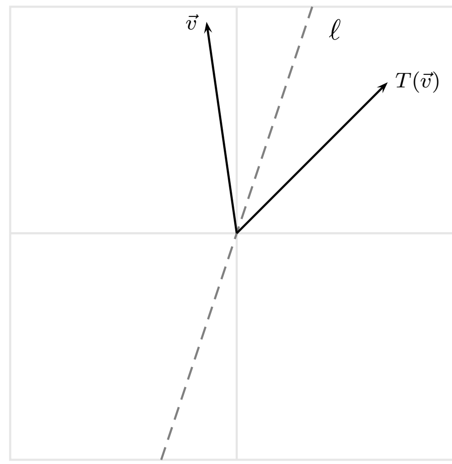
Math 110 Homework Assignment 5

DUE DATE: OCT. 21, 2016

1. In this problem we will practice the idea that, to understand what a linear transformation T does, it is enough to figure out what T does to a special set of vectors and then use the properties of linear transformations to deduce the rest.

Let ℓ be the line of slope 3 in $\mathbb{R}^2$, and let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be "reflection in ℓ". That is, we imagine putting a mirror down along the line ℓ, and for any vector $\vec{v} \in \mathbb{R}^2$, $T(\vec{v})$ is the reflection of $\vec{v}$ in that mirror. The picture at right of a vector $\vec{v}$ and its reflection $T(\vec{v})$ illustrates the idea.



We want to figure out how to compute T. First we need to see that T is a linear transformation. We will do this in the way we analyzed the rotation problem from class, by thinking of the geometric pictures associated to the vector operations, and seeing what happens when we reflect them.

(a) Draw a vector $\vec{v}$ in $\mathbb{R}^2$, the vector $c\vec{v}$ (for some scalar c), and draw $T(\vec{v})$, and $T(c\vec{v})$. In the picture, what does "$T(c\vec{v}) = cT(\vec{v})$" mean? Does it hold for this T?

(b) Draw vectors $\vec{v}$ and $\vec{w}$, and the parallelogram which computes $\vec{v} + \vec{w}$. Then draw the result of applying T to all of this. That is, draw $T(\vec{v})$, $T(\vec{w})$ and $T(\vec{v} + \vec{w})$. In the picture, what does "$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$" mean? Does it hold for this T?

By the results of (a) and (b), T is a linear transformation. To figure out a formula for T, we look for special vectors where it is easy to see how T acts. In this problem, one possible choice are the vectors $\vec{v}_1 = (1, 3)$ and $\vec{v}_2 = (-3, 1)$.

(c) The vector $\vec{v}_1$ lies on the line ℓ. (You can draw the vector to check.) What is $T(\vec{v}_1)$?

(d) The vector $\vec{v}_2$ is orthogonal to the line ℓ. (You *should* draw this to check.) What is $T(\vec{v}_2)$?

(e) Write $(1, 0)$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

(f) Write $(0, 1)$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

(g) Write an arbitrary vector (x, y) as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. (Your answer will involve x and y.)

(h) What is $T(1,0)$?

(i) What is $T(0,1)$?

(j) For an arbitrary vector $\vec{v} = (x, y) \in \mathbb{R}^2$, what is $T(\vec{v})$?

(k) Compute $T(1,5)$ (I chose $(1,5)$ randomly — it does not have any special significance).

(l) Apply T to your answer from (k), that is, compute $T(T(1,5))$.

(m) Your answer from (l) should look familiar. Can you explain geometrically why that happened?

2. For each of the following functions from $\mathbb{R}^2$ to $\mathbb{R}^3$, either show that the function is not a linear transformation (by providing an example where the function does not pass one of the tests), or prove that the function is a linear transformation.

(a) $T(x, y) = (13x + 6y - 2, x + 2y + 5, 3x + 7y - 1)$.

(b) $T(x, y) = (x - y^2, 3x + 4y, 7y)$.

(c) $T(x, y) = (x^2, xy, y^2)$.

(d) $T(x, y) = (x - 2y, 3x + 4y, 7y)$.

3. Fix a vector $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$. For any vector $\vec{v} \in \mathbb{R}^3$, $\vec{u} \times \vec{v}$ is also a vector in $\mathbb{R}^3$, and so we can define a function $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(\vec{v}) = \vec{u} \times \vec{v}$.

Is the function T a linear transformation?