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1. Compute these matrix multiplications:

(a) $\begin{bmatrix} 2 & -1 \\ 5 & 4 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ 7 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 1 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 & 2 \\ -2 & -1 & 3 \\ 7 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 8 & 3 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

2. There are identities among sin and cos concerning angle addition (you may have seen them before). That is, if α and θ are two angles, there is a way to write $\sin(\alpha + \theta)$ in terms of sin's and cos's of α and θ, and the same thing for $\cos(\alpha + \theta)$. There are purely geometric proofs of these identies using triangles, but here's a linear algebra proof:

Let T_α be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is rotation counterclockwise by α, and T_θ the counterclockwise rotation by θ.

(a) Write down the standard matrices for T_α and T_θ, explaining your reasoning for T_α.

(b) Explain what the linear transformation $T_\alpha \circ T_\theta$ does to $\mathbb{R}^2$.

(c) Compute the matrix for $T_\alpha \circ T_\theta$ by multiplying the matrices for T_α and T_θ.

(d) On the other hand, from the description in part (b), you can directly write down the matrix for $T_\alpha \circ T_\theta$. What is that matrix?

(e) Since the matrices from parts (c) and (d) are describe the same linear transformation, they must be equal. What identities among sin and cos must therefore be true?

(f) Using a similar idea, find formulas for $\sin(3\theta)$ and $\cos(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

3. Suppose we have two linear transformations $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by these formulas:

$$T_1(x, y, z) = (7x + 3z, 2x + y + 8z) \quad \text{and} \quad T_2(u, v) = (4u + v, 2u + 3v, -u + 5v)\,.$$

(a) Give the formulas for the composite function $T_3 = T_2 \circ T_1$.

(b) Using these formulas, find the standard matrix C for T_3.

(c) Find the standard matrix A for T_1 and B for T_2.

(d) Compute the matrix product BA showing the details of how you computed the entries. (You should, of course, get matrix C as an answer.)

4. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an invertible linear transformation (i.e., T is a bijection). This means that it's possible to define the inverse function T^{-1}, a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, by

$$T^{-1}(\vec{w}) = \text{the unique vector } \vec{v} \text{ in } \mathbb{R}^n \text{ with } T(\vec{v}) = \vec{w}.$$

That certainly defines a *function* from $\mathbb{R}^n$ to $\mathbb{R}^n$, but we need to check that this function is also a linear transformation. So: Prove that T^{-1} is a linear transformation, using the fact that T is.

Note: For question 4, it might help to first write down (to make you think about it consciously), what the ingredients are for this argument. You know (1) that T is a linear transformation, (2) that T is bijective (and hence injective and surjective), and (3) that T^{-1} is the inverse function for T (and so you know the definition of T^{-1} in terms of T).

This are the only things you know about T, and every step you make will probably involve using one of those properties.

As a perhaps puzzling hint, which you're free to ignore, if you're ever trying to prove that two vectors $\vec{u}_1$ and $\vec{u_2}$ are equal, it's enough to apply T to both sides and check that $T(\vec{u}_1) = T(\vec{u}_2)$, since T is injective.