1. In this problem we will establish a few more relations between the idea of dimension and the ideas of linear dependence and spanning sets. Some of the parts (like (e)) should make intuitive sense. Proving that such statements are actually true is a sign that our definition of dimension is a good one, and also means that we can use these properties freely in the future when we are thinking about subspaces.

Parts (a)–(d) can be solved by using the definition of dimension and the Key Lemma from the class of Thursday, November 17th. Part (f) will likely require the proposition from Tuesday, November 15th.

Suppose that $W \subset \mathbb{R}^n$ is a subspace, that $\dim(W) = d$ (i.e., W is of dimension d), and that $\vec{v}_1, \ldots, \vec{v}_m$ are vectors in W.

(a) Explain why you know that there is a set of d vectors in W which span W.

(b) If $m > d$ prove that $\vec{v}_1, \ldots, \vec{v}_m$ must be linearly dependent.

(c) Explain why you know that there is a set of d vectors in W which are linearly independent.

(d) If $m < d$ prove that $\vec{v}_1, \ldots, \vec{v}_m$ cannot span W.

Now suppose that V is another subspace of $\mathbb{R}^n$, and that $V \subseteq W$.

(e) Prove that $\dim(V) \leqslant \dim(W)$. (SUGGESTION: take a basis for V and use the Key Lemma.)

(f) If $\dim(V) = d$ (i.e., the same dimension as W) prove that $V = W$. (SUGGESTION: Try proving this by contradiction. Assume that $V \neq W$ then start with a basis for V and add to it a vector in W but outside of V. Then use part (b) of the Proposition from Tuesday, November 15th, to get a contradiction.)

(g) Find the mistake in the following argument. Let

$$W = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\right\}.$$

We have seen in class that $(1, 0, -1)$, $(0, 1, -1)$ span W. On the other hand, we know that $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are linearly independent. The Key Lemma says "(# in spanning set)$\geqslant$ (# in linearly independent set)". Applying this to the spanning set $(1, 0, -1)$, $(0, 1, -1)$, and the linearly independent set $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ gives $2 \geqslant 3$, which is clearly false. What went wrong?

NOTES: (**1**) If we take $W = \mathbb{R}^n$ (so that $d = \dim(W) = \dim(\mathbb{R}^n) = n$), then 1(b) becomes the statement that "If we have more than n vectors in $\mathbb{R}^n$, the vectors must be linearly dependent", which was a lemma from the class of Friday, November 11th. That is, the lemma from that class is a special case of 1(b), which tells us what the general statement is for any subspace.

(**2**) As a result of (a) and (b), the number d is the largest size of any set of linearly independent vectors in W. As a result of (c) and (d) the number d is the smallest number of vectors in any set which spans W. Put together, this tells us that the following three numbers are equal :

- The number of elements in a basis for W.
- The smallest number of vectors in any spanning set for W.
- The largest size of any set of linearly independent vectors in W.

This common number is $\dim W$.

2. Let $T\colon \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & 3 & 14 & -3 & 29 & -6 \\ 3 & 3 & 15 & 2 & -3 & 12 \\ 1 & 1 & 5 & 1 & -3 & 5 \\ 2 & 2 & 10 & 4 & -18 & 16 \end{bmatrix}. \text{ The matrix } A \text{ has RREF } \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & 0 & 5 & -1 \\ 0 & 0 & 0 & 1 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(You don't have to prove that this is the RREF.)

(a) Write the third column of A as a linear combination of the first two columns of A.

(b) Write the fifth column of A as a linear combination of the first, second, and fourth columns of A.

(c) Find a basis for $\mathrm{Im}(T)$.

(d) What is $\dim(\mathrm{Im}(T))$?

(e) Is T surjective?

(f) Find a basis for $\mathrm{Ker}(T)$.

(g) What is $\dim(\mathrm{Ker}(T))$?

(h) Is T injective?

(i) What equality does the rank-nullity theorem claim should be true (for this T)?

(j) Is that equality true in this case?