Math 110 Homework Assignment 11

DUE DATE: DEC. 2, 2016

1. The linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is given by the matrix

$$A = \begin{bmatrix} 2 & 0 & 6 & 1 \\ 3 & 1 & 11 & 0 \\ -3 & 0 & -9 & 1 \end{bmatrix}, \text{ which has RREF } \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(You don't have to prove that this is the RREF.)

(a) Find a basis for the image of T.

(b) The vector $\vec{w} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is in the image. Find the linear combination of basis vectors from (a) which gives $\vec{w}$. Use this to find a vector $\vec{v}$ in $\mathbb{R}^4$ with $T(\vec{v}) = \vec{w}$.

(c) Find a basis for the kernel of T.

(d) Write down all the solutions to $T(\vec{v}) = \vec{w}$, with $\vec{w}$ the vector from part (b). You shouldn't have to do any complicated calculations to figure this out.

(e) Find all solutions to the system of equations

$$\begin{array}{rcccccccl} 2x & + & & + & 6z & + & w & = & 5 \\ 3x & + & y & + & 11z & & & = & 5 \\ -3x & & & + & -9z & + & w & = & 0 \end{array}$$

using the "old" method – the RREF method of solving equations that we learned in the first weeks of class.

(f) Explain the connection between (d) and (e).

2. Let $V \subset \mathbb{R}^4$ be the subspace spanned by $\vec{v}_1 = (2, 1, 3, 1)$ and $\vec{v}_2 = (5, 3, 2, 1)$, and set $\vec{w}_1 = (1, 2, 5, 1)$, $\vec{w}_2 = (8, 7, -1, 0)$, $\vec{w}_3 = (3, 1, 2, 2)$.

(a) Is $\vec{w}_2 - \vec{w}_1$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

(b) Does $\vec{w}_1 + V = \vec{w}_2 + V$?

(c) Does $\vec{w}_1 + V = \vec{w}_3 + V$?

3. Suppose that $T_1 \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $T_2 \colon \mathbb{R}^m \longrightarrow \mathbb{R}^p$ are linear transformations. In this question we will practice thinking about compositions by thinking about the kernel of $T_2 \circ T_1$.

(a) Prove that $\vec{v} \in \mathrm{Ker}(T_2 \circ T_1)$ if and only if $T_1(\vec{v}) \in \mathrm{Ker}(T_2)$.

(b) Show that $\mathrm{Ker}(T_1) \subseteq \mathrm{Ker}(T_2 \circ T_1)$.

(c) If T_2 is injective, show that $\mathrm{Ker}(T_1) = \mathrm{Ker}(T_2 \circ T_1)$.

(d) Prove that $T_2 \circ T_1 = 0$ if and only if $\mathrm{Im}(T_1) \subseteq \mathrm{Ker}(T_2)$. (Here 0 means the zero linear transformation – the map that sends each $\vec{v} \in \mathbb{R}^n$ to $\vec{0} \in \mathbb{R}^p$.)

(e) Suppose that $T \colon \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ is a linear transformation, and that $T \circ T = 0$. Prove that $\dim(\mathrm{Im}(T)) \leqslant 3$. (SUGGESTION: Combine (d), the Rank-Nullity theorem, and **H10** Q1(e).)

Note: Here are some basic facts about making arguments about sets. Suppose that X and Y are sets.

- If you are trying to prove that $X \subseteq Y$ then this is the same as showing that for each $x \in X$, it's also true that $x \in Y$. I.e., if, no matter which element $x \in X$ you're given, you can show that this element is also in Y, then you've shown that $X \subseteq Y$.

- If you are trying to prove that two sets X and Y are equal, one way is to first prove that $X \subseteq Y$ and then prove that $Y \subseteq X$. If both of these are true, then $X = Y$. You can try the method above for proving each direction of the containments.

4. In this problem we will continue thinking about compositions, kernels, and invertibility.

(a) If A is an $m \times n$ matrix, explain why $\mathrm{Rank}(A) \leqslant n$ and $\mathrm{Rank}(A) \leqslant m$ (and so $\mathrm{Rank}(A) \leqslant \min(m, n)$).

(b) Let $T \colon \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ be a linear transformation. Prove that $\dim(\mathrm{Ker}(T)) \geqslant 2$.

Suppose that A is a 3×2 matrix, and B is a 2×3 matrix.

(c) Explain why the 3×3 matrix AB can never be invertible. (HINT: what can you say about $\mathrm{Ker}(AB)$?)

(d) Find an example of matrices A and B where the 2×2 matrix BA *is* invertible.

(e) If BA is invertible, what must be the dimension of $\mathrm{Ker}(A)$, and what must be the dimension of $\mathrm{Ker}(B)$? Explain why.