Math 110 — Homework Assignment 14

DUE DATE: JAN. 27, 2016

1. Suppose that $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a multilinear function (but not necessarily alternating), and that we know the following eight values of f:

$$f\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)=e,\quad f\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=\sqrt{7}$$
$$f\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)=0\quad f\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=2,$$
$$f\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)=\sqrt{5},\quad f\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=0,$$
$$f\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)=\pi,\quad f\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=3.$$

Compute these values of f:

(a) $f\left(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}7\\4\end{bmatrix}\right)$ (b) $f\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}1\\5\end{bmatrix}\right)$

(c) $f\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\right)$ (d) $f\left(\begin{bmatrix}-2\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}5\\7\end{bmatrix}\right)$

Leave your answers in symbolic form, i.e., if your answer is $4\pi + 6e - 3 + 2\sqrt{7} - 8\sqrt{5}$, leave it like that instead of writing $13.279020387244307976\ldots$.

2. Suppose that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are vectors in $\mathbb{R}^3$, and we know that $\det_3(\vec{v}_1, \vec{v}_2, v_3) = 4$ and that $\det_3(\vec{v}_1, \vec{v}_4, \vec{v}_3) = 7$.

Compute:

(a) $\det_3(\vec{v}_4, \vec{v}_1, \vec{v}_3)$,

(b) $\det_3(\vec{v}_1, \vec{v}_2 + 3\vec{v}_4, \vec{v}_3)$,

(c) $\det_3(3\vec{v}_2,\ -2\vec{v}_1,\ \vec{v}_3)$,

(d) $\det_3(\vec{v}_1, \vec{v}_2 + 7\vec{v}_3, \vec{v}_3)$, and

(e) $\det_3(\vec{v}_1 + 4\vec{v}_2, \vec{v}_2 - 2\vec{v}_3, 6\vec{v}_1 + 4\vec{v}_3)$.

3. In the definition of the general determinant, we saw that there was only *one* function

$$f : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$$

which was multilinear, alternating, and took the value 1 when we plugged $\vec{e}_1$, $\vec{e}_2$, ..., $\vec{e}_n$ (in that order) into the function.

In that argument, it's important that the function take n vectors in $\mathbb{R}^n$ as input. In this question we're going to see why by making the number of inputs different from the size of the vectors we plug in.

(a) First let's start by making the number of inputs one more than the size of the vectors we plug in. Suppose that we have a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is multilinear (in this case, trilinear) and alternating. If $\vec{v}_1 = (a, b)$, $v_2 = (c, d)$ and $\vec{v}_3 = (e, g)$, compute the only possible value for $f(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. (Compute this by writing each vector $\vec{v}_i$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$ and expanding using the multilinear and alternating rules).

Now let's make the number of inputs one less than the size of the vector we put in. Suppose that $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a function which is multilinear and alternating.

In the definition of the three dimensional determinant, the condition of being multilinear and alternating only needed one more piece of information (that $\det_3(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$) to determine $\det_3$ completely.

(b) In the case of the function f above how many extra pieces of information do we need? For instance, would it be enough just to know the value of $f(\vec{e}_1, \vec{e}_2)$ in order to be able to determine $f(\vec{v}_1, \vec{v}_2)$ for all vectors $\vec{v}_1$, and $\vec{v}_2$ in $\mathbb{R}^3$? What *do* we need to know in order to be able to pin down the function f?