Math 110 Homework Assignment 15

DUE DATE: FEB. 3, 2017

1. Compute the following determinants:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 5 & 9 & 7 \\ 3 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 5 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{vmatrix}$

(d) $\begin{vmatrix} 5 & 2 & 7 & 3 \\ 6 & 2 & 6 & 4 \\ 7 & 4 & 6 & 5 \\ 1 & 6 & 4 & 7 \end{vmatrix}$ (e) $\begin{vmatrix} 3 & 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 5 & 2 & 1 & 7 & 1 \\ 0 & 6 & 0 & 0 & 2 \\ 2 & 6 & 3 & 0 & 1 \end{vmatrix}$ (f) $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 2 & 1 & 5 \end{vmatrix}$

2. Suppose that a, b, c, d, e, and f are numbers such that

$$\begin{vmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{vmatrix} = 7 \text{ and } \begin{vmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{vmatrix} = 11.$$

Find

(a) $\begin{vmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{vmatrix}$ (b) $\begin{vmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{vmatrix}$ (c) $\begin{vmatrix} e & b & 5 \\ f & c & 9 \\ d & a & 1 \end{vmatrix}$

3. The condition that a multilinear function is "alternating" means either of two equivalent conditions is true: (a) if there is a repeated vector, then the output of the function is zero, or (b) swapping two input vectors changes the output by a sign. Why are these conditions equivalent? Let's use this question as a chance to recall the reason.

(a) Suppose that $f\colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a multilinear function, and we know that any time there is a repeated vector in the input the output is zero. Prove that for any v_1, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^3$, $f(\vec{v}_3, \vec{v}_2, \vec{v}_1) = -f(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. (HINT: Consider $f(\vec{v}_1 + \vec{v}_3, \vec{v}_2, \vec{v}_1 + \vec{v}_3)$.)

(b) On the other hand, suppose that swapping any two input vectors changes the output by a sign. Show that for any vectors $\vec{v}$,$\vec{w} \in \mathbb{R}^3$, $f(\vec{v}, \vec{w}, \vec{v}) = 0$.

(This question only asked you to prove the equivalence "in the first and last slots", but hopefully you see how this argument works for any other pair of input positions.)

4. Let A be the 4×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

(a) Write out all the permutations $\sigma \in S_4$ with $\sigma(1) = 4$ (there are six of them).

(b) Using the general formula for $\det(A)$, write out all the terms corresponding to the permuatations you found in part (a).

(c) Use the Laplace expansion formula to expand $\det(A)$ along the first column, and write out all the terms containing a_{41} (there are 18 terms which do not contain a_{41}, and you don't need to write those terms out).

Remarks. There are several motivations for this question. Parts (a) and (b) are simply to make you think about the definition of a permutation, and to understand what the general formula for $\det(A)$ is. Second, your answer for part (c) should be the sum of the terms from part (b). Hopefully it then seems plausible that if we repeated parts (a–b) with all the permuatations such that $\sigma(1) = 1$, $\sigma(1) = 2$, and $\sigma(1) = 3$ (which are the remaining possibilities for permutations in S_4) that we would get the rest of the terms in the Laplace expansion down the first column. This is how one proves the Laplace expansion formula starting from the formula for $\det(A)$. (The case of arbitrary n may seem a bit daunting, but a bit more practice with permutations would let you figure out a general way to organize and carry out the calculation.)