Math 110 Homework Assignment 17

DUE DATE: FEB. 17, 2017

1. Suppose that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is given (in the standard coordinates) by the matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix}.$$

Let $\mathcal{B}$ be the basis $\mathcal{B} = [(1,1,1),(2,0,1),(3,2,1)]$ of $\mathbb{R}^3$, and let $\mathcal{A}$ be the basis $\mathcal{A} = [(3,5),(1,2)]$ of $\mathbb{R}^2$, find the matrix for $T$ with respect to the new basis on both sides.

2. Let $\vec{v}_1 = (1,2)$, $\vec{v}_2 = (2,-1)$, and let $\mathcal{B}$ be the basis $\mathcal{B} = [\vec{v}_1, \vec{v}_2]$ of $\mathbb{R}^2$.

Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = \vec{0}$.

(a) Write down the matrix for $T$ in the new basis $\mathcal{B}$. (You should be able to do this directly from the definition of $T$ and the definition of "writing the matrix of a linear transformation with respect to a basis").

(b) Use this to write down the matrix for $T$ in the standard basis.

You might want to compare the answer for (b) with the answer for **H6** 3(a), with $m = 2$. Can you see why these are the same?

NOTE: In part (b) you need to go from the matrix in $\mathcal{B}$-basis form to the matrix in standard basis form, which is the reverse of what we did in class, so think for a bit to figure out which way the change of basis matrices should go.

3. We'll check in class that the determinant of a square matrix doesn't change when we change basis. The purpose of this question is to show that the *trace* of a matrix also doesn't change when we change basis.

For an $n \times n$ matrix $A$, the *trace* of $A$, $\mathrm{tr}(A)$ is the sum of the numbers on the diagonal. For instance, if

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 6 & 0 & 4 \end{bmatrix}$$

then $\mathrm{tr}(A) = 1 + 7 + 4 = 12$.

In the $a_{ij}$ notation for the entries of a matrix, $\mathrm{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$.

(a) If $A$ and $B$ are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (the formula in the book for $ij$-th entry for a product of matrices may help).

(b) Suppose that $A$ is an $n \times m$ matrix and $B$ an $m \times n$ matrix. Then both products $AB$ (an $n \times n$ matrix) and $BA$ (an $m \times m$ matrix) are square matrices, so we can take their traces. PROVE OR DISPROVE: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ in this case.

(c) If $C$ is an $n \times n$ matrix, and $M$ an invertible $n \times n$ matrix, prove that

$$\mathrm{tr}(C) = \mathrm{tr}(M^{-1}CM).$$

SUGGESTION: If you make the right choice of matrices $A$ and $B$, part (c) will follow from part (a) with very little work.

4. Suppose that $x_1, x_2, \ldots, x_n$ are numbers. The $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ x_1^3 & x_2^3 & x_3^3 & \cdots & x_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

is called the *Vandermonde matrix*, and is surprisingly useful to know a few basic facts about it. Let's establish one of them now.

If any of the two $x_i$'s are equal to each other, then of course $\det(A) = 0$ since we will have two repeated columns. What we'd like to show is that if all of the $x_i$'s are different, then $\det(A) \neq 0$, i.e., that $A$ is an invertible matrix.

(a) Explain why showing that $\det(A) \neq 0$ is the same as showing that $\det(A^t) \neq 0$, where $A^t$ means the transpose of $A$. [This is a very short answer].

(b) Suppose that $A^t$ is invertible. Explain why this means that $\mathrm{Ker}(A^t) = \{\vec{0}\}$.

(c) Conversely, suppose that $\mathrm{Ker}(A^t) = \{\vec{0}\}$. Explain why this means that $A^t$ is invertible. (SUGGESTION: The rank-nullity theorem may help at one point.)

(d) Explain why showing that $\det(A^t) \neq 0$ is the same as showing that the only vector in $\mathrm{Ker}(A^t)$ is the zero vector.

(e) If $\vec{v} = (c_0, c_1, \ldots, c_{n-1})$ is a vector in the kernel of $A^t$, and $\vec{v}$ is not the zero vector, explain how this would give you a polynomial of degree $\leqslant n-1$ with more than $n-1$ roots, which would be a contradiction. [HINT: write out what $A^t\vec{v} = \vec{0}$ means.] Make sure that you answer carefully, for instance, in your explanation why is it important that all the $x_i$'s be different?