Math 110 Homework Assignment 18

DUE DATE: MAR. 3, 2017

1. The FIBONACCI NUMBERS are the numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$. (So for instance $F_2 = F_1+F_0 = 1+0 = 1$, $F_3 = F_2+F_1 = 1+1 = 2$, and $F_4 = F_3 + F_2 = 2 + 1 = 3$, etc.)

(a) Suppose that we set $\vec{w}_n = (F_n, F_{n-1})$ for $n \geq 1$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Show that the recursion relation above means that $\vec{w}_{n+1} = A\vec{w}_n$.

(b) Use the eigenvectors of A to find a formula for $A^k\vec{w}_1$ for any $k \geq 0$.

(c) Use the answer from (b) to find a formula for the n-th Fibonacci number F_n.

(d) The LUCAS NUMBERS are the numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Find a formula for the n-th Lucas number.

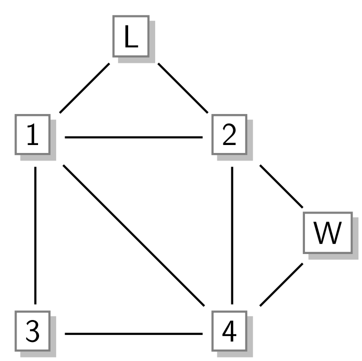
2. Let A be the matrix

$$A = \begin{bmatrix} 19 & -14 \\ 21 & -16 \end{bmatrix}.$$

Find a formula for the entries of A^k (for $k \geq 1$). As a check, compute A^2 and A^3 and to see if they match your formulas.

HINT: The first column of any 2×2 matrix is "where $(1, 0)$ gets sent", therefore if $\vec{w}_0 = (1, 0)$, then the first column of A^k is $A^k\vec{w}_0$.

3. The diagram at right shows a simple Win/Lose game. In each turn, we are at one of positions 1, 2, 3, or 4, or we have either won (W) or lost (L). If we are at position W or L we stay there since the game is over. If we're at any of the positions 1 through 4, then during that turn we leave (with equal probability) along one of the lines coming out of that position. In other words, if we're at position 1, then there's a 1/4 chance that we'll end up at L the next turn, a 1/4 chance of ending up at 2, a 1/4 chance of ending up at 3, and a 1/4 chance of ending up at 4.



On the other hand, if we're at position 3, then there's a 1/2 chance of ending up at position 1 and a 1/2 chance of ending up at position 4 on the next turn.

(a) Write down the 6×6 transition matrix that tells us how to get from one turn of the game to another. When writing down the matrix, let's use the order of positions 1, 2, 3, 4, W, and then L.

If you write down the correct matrix, the eigenvectors should be: $\vec{v}_1 = (1, -1, -1, 1, 0, 0)$, $\vec{v}_2 = (3, 2, 2, 3, -5, -5)$, $\vec{v}_3 = (0, -4, 2, 0, 1, 1)$, $\vec{v}_4 = (5, 0, 0, -5, 1, -1)$, $\vec{v}_5 = (0, 0, 0, 0, 1, 0)$, and $\vec{v}_6 = (0, 0, 0, 0, 0, 1)$.

(b) For each of the squares 1, 2, 3, and 4, work out the probability of winning if you start on that square.

(c) Which starting square has the best chance of winning?

The matrix

$$\begin{bmatrix} 1 & 3 & 0 & 5 & 0 & 0 \\ -1 & 2 & -4 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & -5 & 0 & 0 \\ 0 & -5 & 1 & 1 & 1 & 0 \\ 0 & -5 & 1 & -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{10} & \frac{1}{15} & \frac{2}{15} & \frac{1}{10} & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & -\frac{1}{10} & 0 & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{5} & 1 & 0 \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & 0 & 1 \end{bmatrix}$$

may be useful in answering the question.