1. Suppose that A_n is the $n \times n$ matrix which has 2's on the diagonal, and 1's everywhere else:

$$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},\ A_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \ldots$$

and suppose that B_n is the $n \times n$ matrix which is just filled with minus-ones:

$$B_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\ B_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix},\ B_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \ldots$$

In this problem we will use some of our knowledge of characteristic polynomials to find a formula for $\det(A_n)$.

(a) Explain why $\det(A_n) = \det(I_n - B_n)$, where I_n is the $n \times n$ identity matrix.

(b) If $P_n(t)$ is the characteristic polynomial of B_n, explain why $\det(A_n) = P_n(1)$.

This means that we can compute $\det(A_n)$ by first figuring out the characteristic polynomial of B_n and then plugging in a value. It might seem like more work to compute the characteristic polynomial of B_n, but

(c) Since B_n has rank 1, explain why this means that t^{n-1} has to divide $P_n(t)$. (HINT: How big is the kernel of B_n? What is the relation between the kernel of B_n and the eigenspace E_0 for B_n?).

This means that $P_n(t)$ is of the form $t^{n-1}(t-a)$ for some number a.

(d) Either by looking at the trace of B_n, or by seeing what happens to the vector $\vec{v} = (1, 1, \ldots, 1)$ of all 1's when you put it through B_n, find the value of a.

(e) What is $\det(A_n)$?

(f) What is the determinant of the $n \times n$ matrix C_n which has 5's on the diagonal, and 1's everywhere else?

2. For the following three matrices, find their characteristic polynomials, the algebraic and geometric multiplicities of each eigenvalue, and a basis for each of their eigenspaces.

(a) $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -6 & 9 & 6 \\ 0 & 3 & 0 \\ -12 & 12 & 11 \end{bmatrix}$

To make factoring the characteristic polynomials a bit easier, 2 is a root of each one.

(d) Which of the matrices above is diagonalizable?

(e) For each of the matrices A from part (d), find an invertible matrix N so that $N^{-1}AN$ is a diagonal matrix.

3. Suppose that B and C are $n \times n$ matrices, and that N is an invertible $n \times n$ matrix so that $C = N^{-1}BN$.

(a) Find a formula expressing B in terms of N and C (i.e., "Solve for B").

(b) Show that $C^2 = N^{-1}B^2N$. (Hint: just multiply.)

(c) Suppose that D is a diagonal matrix with real entries. If we want to find a diagonal matrix C with real entries, such that $C^2 = D$, what has to be true about the eigenvalues of D?

(d) Let $A = \begin{bmatrix} -16 & -10 \\ 50 & 29 \end{bmatrix}$.

Find a real matrix B with $B^2 = A$ (i.e., a "square root" of A).

Last minute hint for (d): Combining (a)–(c) they add up to a suggestion. To find a square root of A, first change basis to diagonalize A, then take a square root of the diagonal matrix, then change basis back.