1. In class we saw that if an $n \times n$ matrix A has a dominant eigenvalue, and starting with a "sufficiently general" vector $\vec{w}$, then as $k \to \infty$ the vectors $A^k\vec{w}$ would approach the direction of an eigenvector for the dominant eigenvalue. As we do this $A^k\vec{w}$ may get very large (or very small if the dominant eigenvalue has magnitude less than 1).

This gives us a potential numerical algorithm for finding an eigenvector for the dominant eigenvalue. Start with a $\vec{w}$ and see what direction $A^k\vec{w}$ is approaching as $k \to \infty$. In order to deal with the "$A^k\vec{w}$ gets large" (or "$A^k\vec{w}$ gets small") problem, one thing we could do is to rescale the vector as we go so as to keep its size reasonable. One way to do this is to always rescale the vector so that its size is 1. An alternate way (which avoids a lot of square roots) is to pick one coordinate and rescale it so that it is equal to 1. That is, we could try and repeat the procedure :

> "Compute $A\vec{w}$, and then scale the answer so that one of the coordinates is 1."

and then take the result and do the same thing (and then the same thing again, etc).

Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix} \quad \text{and} \quad \vec{w} = (0,1,0).$$

In this problem let's always rescale so that the first coordinate is 1. The first two steps of this procedure are :

$$(0,1,0)\underbrace{\xrightarrow{A}(-2,0,-3)\xrightarrow{\text{scale by } -\frac{1}{2}}(1,0,\tfrac{3}{2})}_{\text{1st step}}\underbrace{\xrightarrow{A}(4,2,\tfrac{13}{2})\xrightarrow{\text{scale by } \frac{1}{4}}(1,\tfrac{1}{2},\tfrac{13}{8})}_{\text{2nd step}}\xrightarrow{A}\cdots$$

(a) Compute the next three steps of this procedure, showing your work.

(b) Compute the result after 5 more steps (you don't have to show the intermediate calculations — just the answer).

(c) What vector does this seem to be approaching?

(d) Check that your answer in (c) is an eigenvector of A. What is its eigenvalue?

(e) Find the eigenvalues of A and check that your answer in (d) is the dominant eigenvalue.

NOTE: In parts (a) and (b) (and in question 2) you are welcome to use a computer to help in your computations.

2. The previous question claimed that if we start with a "sufficiently random" vector $\vec{w}$, and repeat the procedure

> "Compute $A\vec{w}$, and then scale the answer so that one of the coordinates is 1."

then the result (after repeating this a bunch of times) would converge on an eigenvector for the dominant eigenvalue of A.

The purpose of this question is to figure out what "sufficiently random" means, and why it is there. We'll use the matrix

$$B = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}.$$

(a) Starting with the vector $\vec{w}_1 = (1, 0, 0)$, do this procedure 10 times (you don't need to show all the calculations, just show the first two and the final answer. This time you should re-scale so that the last coordinate is 1).

(b) What vector $\vec{v}_1$ does this seem to be approaching?

(c) Check that this vector from (b) *is* an eigenvector of B and find its eigenvalue.

(d) Find the eigenvalues for B (all are integers). What is the dominant eigenvalue? Does this seem to contradict the statement above and part (c)?

(e) Starting with the vector $\vec{w}_2 = (2, 0, 1)$, again repeat the procedure 10 times (again you don't need to show all the work).

(f) What vector $\vec{v}_2$ does this seem to be approaching? Check that it is an eigenvector of B and find the corresponding eigenvalue.

(g) Find all the eigenvectors of B, and write $\vec{w}_1$ and $\vec{w}_2$ as linear combinations of those eigenvectors.

(h) Use the answer for (g) to explain what happened in (c), and why things worked in (f).

(i) If we have a diagonalizable $n \times n$ matrix B, and we start with a vector $\vec{w}$ and repeat the procedure above, what condition on $\vec{w}$ will ensure that it converges to an eigenvector for the dominant eigenvalue of B? Will this condition be true for "most" vectors $\vec{w}$ in $\mathbb{R}^n$?

3. Draw the Gerschgorin discs for the following matrices

$$\text{(a)} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} i & 0 & i \\ 0 & -i & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 3i & i & -1 \\ -1 & 4 & 1 \\ 2 & -1 & -4+i \end{bmatrix}$$

For the matrices in (a) and (b), find the eigenvalues and include them in the pictures.

4. Let A be an $n \times n$ matrix, whose entries could be real or complex numbers. Suppose that the diagonal entries of A are much larger than the size of the other entries in the same row, in the sense that for each row i

$$2\|a_{ii}\| > \sum_{j=1}^{n} \|a_{ij}\| = \|a_{i1}\| + \|a_{i2}\| + \|a_{i3}\| + \cdots + \|a_{in}\|.$$

Prove that A is invertible.

The matrix in 3(d) satisfies this condition, if you want an example to look at.

(HINT: How is A *not* being invertible connected to the eigenvalues of A?) Note that the sum on the right side of the inequality includes the term $\|a_{ii}\|$.