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1. In **H18** Q1 we found a formula for the n-th Fibonacci number. In this problem we will find the same formula in a different way. Let $W \subset \mathbb{R}^\infty$ be the subset

$$W = \left\{ (x_1, x_2, x_3, x_4, \ldots) \mid x_{n+2} = x_{n+1} + x_n \text{ for all } n \geqslant 1 \right\}.$$

By **H22** 1(d) we know that W is a subspace of $\mathbb{R}^\infty$.

(a) Suppose that $v = (x_1, x_2, x_3, x_4, \cdots)$ and $w = (y_1, y_2, y_3, \cdots)$ are vectors in W, and that $x_1 = y_1$ and $x_2 = y_2$. Prove that $v = w$. (In other words, show that a vector in W is completely determined by its first two coordinates.)

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. These are the two roots of the polynomial $t^2 - t - 1$.

(b) Show that the vectors $u_1 = (1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots)$ and $u_2 = (1, \beta, \beta^2, \beta^3, \ldots)$ are in W.

(c) Show that $(1, \alpha)$ and $(1, \beta)$ are linearly independent in $\mathbb{R}^2$.

(d) Show that $(1, \alpha)$ and $(1, \beta)$ span $\mathbb{R}^2$. (You can deduce this from (c) with very little work.)

(e) Combining (a), (c), and (d) show that u_1 and u_2 are a basis for W.

Now let $v \in W$ be the "Fibonacci vector", $v = (F_0, F_1, F_2, F_3, \ldots) = (0, 1, 1, 2, 3, 5, \cdots)$. From the definition of the Fibonacci numbers, v is a vector in W. Therefore, by part (e) v is a linear combination of u_1 and u_2.

(f) Write v as a linear combination of u_1 and u_2. (Do this, as in your proof in (e), by matching the first two coordinates.)

(g) Comparing the entries of v and the that of the linear combination from (f), what formula for the Fibonacci numbers do we get?

2. Let $V = P_2$ the polynomials of degree $\leqslant 2$, and define a linear transformation $T \colon V \longrightarrow V$ by $T(p) = (x-1)p'$, where p' is the derivative of p. So, for example, $T(3x - 5) = (x - 1)(3 - 0) = 3x - 3$

(a) Where does T send $p = 2x^2 - 5x + 6$?

(b) Write the matrix for T with respect to the basis $\mathcal{B} = [1, x, x^2]$ (where $\mathcal{B}$ is the basis for both input and output).

(c) What is $2x^2 - 5x + 6$ in $\mathcal{B}$-coordinates? Put this vector through your answer for (b). How does this compare to your answer from (a)?

Note: What part (c) (and the rest of the question) is emphasizing is the purpose of a "matrix with respect to a given basis". If we write the input vectors in $\mathcal{B}$-coordinates, the result of putting this vector through the matrix is (in $\mathcal{B}$-coordinates) the same as the result of applying T.

(d) Find the eigenvalues and eigenvectors for the matrix in (b).

(e) What polynomials in V correspond to the eigenvectors from (d)? Compute what T does to each of these polynomials to check that you get what you expect.

3. This question explores a general method for showing that a set of functions are linearly independent. We will only look at the case of three functions, but the idea and the arguments work for any number of functions.

Given three functions f_1, f_2, and $f_3 \in C^\infty(\mathbb{R})$, we define their *Wronskian* to be

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

where f' and f'' mean the first and second derivative of f, and the vertical bars mean to take the determinant. Note that the Wronskian (the determinant of the matrix above) is a function.

What we want to prove is this: If $W(f_1, f_2, f_3) \neq 0$ (i.e., is not the zero function) then f_1, f_2, and f_3 are linearly independent.

(a) To get a feeling for the definition, compute $W(x, \sin(x), \cos(x))$.

Now let $f_1, f_2, f_3 \in C^\infty(\mathbb{R})$ be any functions, and let's try and prove the statement above.

(b) Explain why, if $W(f_1, f_2, f_3) \neq 0$, there must be an $x_0 \in \mathbb{R}$ such that $W(f_1, f_2, f_3)(x_0) \neq 0$.

(c) Explain why, for this value of x_0,

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) & f_3(x_0) \\ f_1'(x_0) & f_2'(x_0) & f_3'(x_0) \\ f_1''(x_0) & f_2''(x_0) & f_3''(x_0) \end{vmatrix} \neq 0.$$

(d) Now suppose that $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$. Differentiate this relation twice, and plug x_0 into each of the three relations to get three equations for c_1, c_2, and c_3.

(e) Explain why the only solution to the equations in (d) is $c_1 = c_2 = c_3 = 0$, and hence that f_1, f_2, f_3 are linearly independent.