DUE DATE: OCT. 5, 2017

1. Let $\mathbf{F}(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the function $\mathbf{F}(x,y) = (e^{x^2},\, y\sin(\pi x),\, xy)$, and $\mathbf{G} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the function $\mathbf{G}(u,v,w) = (\cos(uv), w - u^2)$.

(a) Compute $\mathbf{F}(1,1)$, and let q be this point in $\mathbb{R}^3$.

(b) Write out the composite function $\mathbf{G} \circ \mathbf{F}$, and compute directly $\mathbf{D}(\mathbf{G} \circ \mathbf{F})(1,1)$.

(c) Compute $\mathbf{DF}(1,1)$ and $\mathbf{DG}(q)$.

(d) Compute the product $\mathbf{DG}(q)\,\mathbf{DF}(1,1)$ and verify the chain rule in this case.

2. Let $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function $\mathbf{F}(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Note that $\mathbf{F}(2, \frac{5\pi}{6}) = (-\sqrt{3}, 1)$.

(a) Compute $\mathbf{DF}(2, \frac{5\pi}{6})$.

Suppose that $\mathbf{G}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function, and define a new function $\mathbf{H}$ by $\mathbf{H} = \mathbf{G} \circ \mathbf{F}$.

(b) What formula does the chain rule give for computing $\mathbf{DH}(2, \frac{5\pi}{6})$ in terms of $\mathbf{DG}$ and $\mathbf{DF}$?

(c) Suppose we know that $\frac{\partial \mathbf{H}}{\partial r}(2, \frac{5\pi}{6}) = 2$ and $\frac{\partial \mathbf{H}}{\partial \theta}(2, \frac{5\pi}{6}) = 3$. Use your answer from (b) to find $\frac{\partial \mathbf{G}}{\partial x}(-\sqrt{3}, 1)$ and $\frac{\partial \mathbf{G}}{\partial y}(-\sqrt{3}, 1)$. (This will involve inverting a matrix.)

(d) Suppose we are at the point $(-\sqrt{3}, 1)$. In what direction $\vec{v}$ should we go so that the instantaneous rate of change of $\mathbf{G}$ through $(-\sqrt{3}, 1)$ in the direction $\vec{v}$ is zero?

3. Define the function $\mathbf{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\mathbf{F}(x,y) = \begin{cases} \frac{\cos(3x+2y)-1}{3x+2y} & \text{if } 3x+2y \neq 0 \\ 0 & \text{if } 3x+2y = 0 \end{cases}.$$

Arguments similar to that of **H1 Q5(b)** show that $\mathbf{F}$ is a continuous function on $\mathbb{R}^2$. In this problem we will show that $\mathbf{F}$ is a differentiable function, and compute $\mathbf{DF}$ at the point $(2, -3)$. To start with, we define the function $\mathbf{G}\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathbf{G}(u) = \begin{cases} \frac{\cos(u)-1}{u} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}.$$

(a) The derivative in one variable is defined by a limit. Set up the limit which computes the derivative of $\mathbf{G}$ at $u = 0$.

(b) Evaluate the limit in part (a). (You may use L'Hôpital's rule if you wish.)

(c) Find a function $\mathbf{H}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\mathbf{F} = \mathbf{G} \circ \mathbf{H}$.

(d) By applying the chain rule, establish that $\mathbf{F}$ is a differentiable function on $\mathbb{R}^2$.

(e) Use the chain rule to calculate $\mathbf{DF}(2, -3)$.

4.

(a) Describe and sketch the graph $z = \dfrac{1}{x^2 + y^2}$. (Over $\mathbb{R}^2 \setminus \{(0,0)\}$.)

(b) Show that the parameterization $(x(t),\, y(t),\, z(t)) = (e^t \cos(t),\, e^t \sin(t),\, e^{-2t})$. lies on the graph from part (a).

(c) Describe what this curve does, and sketch it on the graph from part (a).