Math 280 — Homework Assignment 4

DUE DATE: OCT. 12, 2017

1. *Spirals*
Consider the parameterized curve given by $x(t) = e^t \cos(t)$ and $y(t) = e^t \sin(t)$ for $t \in \mathbb{R}$.

(a) Find the length of the curve between the times $t = 0$ and $t = 5$.

(b) For this curve, prove that at any point the vector joining that point to the origin meets the tangent line at that point in a constant angle (i.e., an angle independent of t). Also find the angle.

(c) (BONUS QUESTION) Suppose that we have a parameterization of a curve of the form $x(t) = r(t)\cos(t)$ and $y(t) = r(t)\sin(t)$, with the property similar to the property in part (b): for any point of the curve the vector joining the origin to that point meets the tangent line to that point at a constant angle θ. Show that $r(t)$ must be of the form $r(t) = e^{kt}$, and find the constant k in terms of θ.

2. *Velocity and acceleration*

(a) Let $\vec{u}(t) = (u_1(t), u_2(t))$ be a differentiable function $\mathbb{R} \longrightarrow \mathbb{R}^2$. Prove the formula $\frac{d}{dt}\|u(t)\|^2 = 2\,u(t) \cdot u'(t)$.

(b) If $(x(t), y(t))$ is a parameterized curve in $\mathbb{R}^2$, show that the speed of this parameterization is constant if and only if the acceleration is always perpendicular to the velocity vector.

(c) If an object (like a planet) orbits around a more massive object (like the sun) the orbit will be an ellipse with the massive object at one of the two foci of the ellipse.

The parameterization $x(t) = 2\cos(t)$ and $y(t) = \sin(t)$ is a parameterization of the ellipse $\frac{x^2}{4} + y^2 = 1$, which has foci at the points $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Could this parameterization be a parameterization of an object in orbit? Explain why or why not. (NOTE: Part (c) is unconnected with (a) or (b).)

3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = xy + z^2$ and C the parameterized curve given by $x(t) = 3t^2$, $y(t) = t\sin(t)$, and $z(t) = e^{2t}$.

(a) Let p be the point on C corresponding to $t = \pi$, and $\vec{v}$ the velocity vector at that point. Find p and $\vec{v}$.

(b) Find the gradient ∇f at the point p.

(c) Compute $\nabla f(p) \cdot \vec{v}$.

(d) Write out the composite function $f(x(t), y(t), z(t))$ and compute its derivative when $t = \pi$.

(e) The answers to (c) and (d) are of course the same. Explain why this is a consequence of the chain rule and the definition of the gradient.

4. For the following vector fields, identify those which are conservative, and those which are not conservative. For those which are conservative, find a potential function $\mathbf{F}$. For those which are not conservative, explain how you know this.

(a) $\mathbf{G}(x, y, z) = (y^2,\ 2xy + y^2,\ 2yz)$.

(b) $\mathbf{G}(x, y, z) = (x^2 + y^2,\ 2x^3 + 1,\ z^2)$.

(c) $\mathbf{G}(x, y, z) = (\cos(y) - z\cos(x),\ -x\sin(y),\ -\sin(x))$.

(d) $\mathbf{G}(x, y, z) = (\sin(z) - y\sin(x),\ \cos(x),\ x\sin(z))$.

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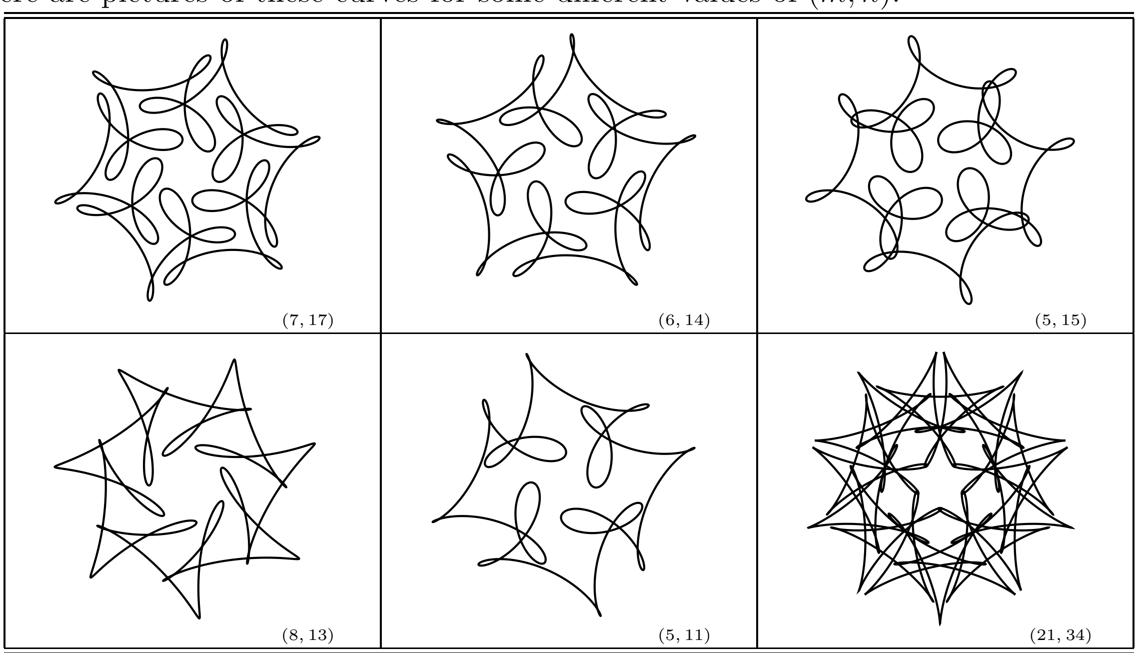
Given integers m and n, define a parameterized curve by

$$\mathbf{c}(t) = \left(\cos(t) + \tfrac{1}{2}\cos(mt) + \tfrac{1}{3}\sin(nt),\ \sin(t) + \tfrac{1}{2}\sin(mt) + \tfrac{1}{3}\cos(nt)\right), \quad t \in [0, 2\pi].$$

Here are pictures of these curves for some different values of (m, n).



There is nothing mathematically special about these curves, but they are elegant. They are taken from the book *Creating Symmetry*, by Frank Farris.