1. Let f be a function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$, and $\mathbf{F}$ and $\mathbf{G}$ vector fields on $\mathbb{R}^3$ (i.e, functions $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$). State whether each of the following expressions is a function from $\mathbb{R}^3$ to $\mathbb{R}$, a vector field, (i.e., a function from $\mathbb{R}^3$ to $\mathbb{R}^3$), or meaningless.

(a) $\mathrm{grad}(\mathrm{grad}(f))$ (b) $\mathrm{Curl}(\mathrm{grad}(f)) - \mathbf{F}$ (c) $\mathrm{Curl}(\mathrm{Curl}(\mathbf{F})) - \mathbf{G}$

(d) $\mathrm{Curl}(\mathbf{F}) \cdot \mathbf{G}$ (e) $\mathrm{Div}(\mathrm{Div}(\mathbf{F}))$ (f) $\mathrm{Div}(\mathrm{Curl}(\mathrm{grad}(f)))$

2. Compute Div and Curl for the following vector fields:

(a) $\mathbf{F}(x, y, z) = (x,\ y,\ z)$ (b) $\mathbf{F}(x, y, z) = (yz,\ xz,\ xy)$

(c) $\mathbf{F}(x, y, z) = (3x^2y,\ x^3 + y^3,\ z^4)$ (d) $\mathbf{F}(x, y, z) = (e^x \cos(y) + z^2,\ e^x \sin(y) + xz,\ xy)$

3. The vector field $\vec{r}(x, y, z) = (x, y, z)$ is commonly used in physics. Let a, b, $c \in \mathbb{R}$ be any real numbers, and set $\vec{w} = (a, b, c)$.

(a) Compute $\vec{r} \times \vec{w}$.

(b) Compute the curl of your answer from (a).

4.

(a) Is there a vector field $\mathbf{F}$ such that $\mathrm{Curl}(\mathbf{F}) = (xy^2,\ yz^2,\ zx^2)$? Explain.

(b) Is there a vector field $\mathbf{F}$ so that $\mathrm{Curl}(\mathbf{F}) = (2, 1, 3)$? If so, find one.

5. A vector field $\mathbf{F}$ is called *incompressible* if $\mathrm{Div}(\mathbf{F}) = 0$, and *irrotational* if $\mathrm{Curl}(\mathbf{F}) = 0$.

(a) Show that any vector field of the form $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$ is irrotational.

(b) Show that any vector field of the form $\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$ is incompressible.

(c) Find constants a, b, and c so that the vector field $\mathbf{F}(x, y, z) = (3x - y + az,\ bx - z,\ 4x + cy)$ is irrotational. For these values of a, b, and c, find a function f with $\nabla f = \mathbf{F}$.

The *Laplacian* of a function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined to be $\mathrm{Div}(\mathrm{grad}(f))$, and is denoted Δf. (NOTE: Δ may look a bit like the gradient, ∇, but it isn't – it's the Laplacian.) Composing the gradient and the divergence, we compute that for a function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\Delta f = \frac{\partial^2 f}{\partial^2 x_1} + \frac{\partial^2 f}{\partial^2 x_2} + \cdots + \frac{\partial^2 f}{\partial^2 x_n}.$$

A function f is called *harmonic* if $\Delta f = 0$.

(d) If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ which is both incompressible and irrotational, show that $\mathbf{F}$ is the gradient of a harmonic function f.