DUE DATE: NOV. 30, 2017

1. Let T be the region in $\mathbb{R}^3$ described by $\sqrt{x^2+y^2} \leqslant z \leqslant 4$ and S its boundary surface, oriented outwards (outwards from T that is). Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = (xz, yz, z).$$

(a) Sketch T. How many pieces does the boundary surface S have?

(b) Work out the flux integral $\iint_S \mathbf{F} \cdot dS$ directly by parameterizing each of the pieces of S and computing the integrals of each of them.

(c) Compute $\mathrm{Div}(\mathbf{F})$ and compute (by working it out) the integral of $\mathrm{Div}(\mathbf{F})$ over T. This answer should be the same as (b), by the Divergence theorem.

2. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = (y, z, x^2)$ on $\mathbb{R}^3$.

(a) Let S_1 be the top half of the unit sphere, oriented upwards. Compute $\iint_{S_1} \mathbf{F} \cdot dS$.

(b) Let S_2 be the disk $x^2 + y^2 \leqslant 1$ in the xy-plane, oriented upwards. Compute $\iint_{S_2} \mathbf{F} \cdot dS$.

(c) Find a vector field $\mathbf{G}$ with $\mathrm{Curl}(\mathbf{G}) = \mathbf{F}$.

(d) Let $\mathbf{c}$ be the unit circle $x^2+y^2 = 1$ in the xy-plane, oriented counterclockwise when viewed from the positive z-axis. Notice that $\mathbf{c}$ is the oriented boundary curve of both S_1 and S_2. Compute $\int_{\mathbf{c}} \mathbf{G} \cdot ds$.

(e) In general, if S_1 and S_2 are two oriented surfaces in $\mathbb{R}^3$, which have the same oriented boundary curve $\mathbf{c}$, and if $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ with $\mathrm{Div}(\mathbf{F}) = 0$, explain why $\iint_{S_1} \mathbf{F} \cdot dS = \iint_{S_2} \mathbf{F} \cdot dS$.

There are at least two possible explanations. One is to use Stokes' theorem and the ideas above, and another to use the divergence theorem. Don't forget to explain how the assumption that $\mathrm{Div}(\mathbf{F}) = 0$ enters into either argument.

3. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

(a) Let $\mathbf{c}_1$ be the unit circle $x^2+y^2=1$ in the xy-plane, oriented counterclockwise when viewed from above. Compute $\int_{\mathbf{c}_1} \mathbf{F}\cdot ds$.

(b) Let $\mathbf{c}_2$ be the unit circle $x^2+y^2=1$ in the plane $z=3$, oriented counterclockwise when viewed from above. Compute $\int_{\mathbf{c}_2} \mathbf{F}\cdot ds$.

(c) Compute that $\mathrm{Curl}(\mathbf{F}) = (0,0,0)$.

(d) Explain how to use Stokes's theorem to deduce that the integrals in (a) and (b) are the same. (SUGGESTION: Consider the surface $x^2+y^2=1$, $0 \leqslant z \leqslant 3$. Pay attention to the orientation of the surface and the boundary.)

(e) Let $\mathbf{c}_3$ be the circle $(x-3)^2+y^2=1$ in the xy-plane, oriented counterclockwise when viewed from above. By considering the disk $(x-3)^2+y^2 \leqslant 1$, $z=0$, show that $\int_{\mathbf{c}_3} \mathbf{F}\cdot ds = 0$.

(f) Why can't we use an argument like (d) to conclude that that the integrals in (a) and (b) are also zero?

NOTE: Question 4 is an example of the fact that integrals along special kinds of vector fields can detect topological (i.e., "shape") information. (We've seen other examples of this kind of behaviour in class.) In question 4, integrating $\mathbf{F}$ around any closed curve $\mathbf{c}$ detects how many times $\mathbf{c}$ winds around the z-axis.

Orientation inducing caveman (shown at right)[†] may help in making sure the surface and boundary curves are oriented compatibly when applying Stokes's theorem.

[†]Picture courtesy Adina Bogart O'Brien