DUE DATE: DEC. 7, 2017

1. Let $\mathbf{c}$ be the top half of the unit circle, oriented from $(-1,0)$ to $(1,0)$, and $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = (\ln(x+5) + y^2, 2xy - y^2).$$

Check that $\mathrm{Curl}_2(\mathbf{F}) = 0$ and use the flexibility theorem for curves to compute the integral $\displaystyle\int_{\mathbf{c}} \mathbf{F} \cdot ds$. Flexing the curve to the line joining $(-1,0)$ and $(1,0)$ is a possibility.

2. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = (ye^{z^2} - xe^{xy},\ ye^{xy} + \tan(z^2+z+1),\ x^2).$$

Check that $\mathrm{Div}(\mathbf{F}) = 0$, and use the flexibility theorem for surfaces to compute $\displaystyle\iint_S \mathbf{F} \cdot dS$ where $S$ is the top half of the unit sphere oriented outwards. ("Flexing" it to the unit disk in the $xy$-plane seems like a good bet.)

3. Let $\mathbf{c}_1$ be the top half of the unit circle, oriented counterclockwise, $\mathbf{c}_2$ be the line segment joining $(1,0)$ to $(-1,0)$, oriented from $(1,0)$ to $(-1,0)$, and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = (x^2 - y,\ xy - \arcsin(y) + e^{y^3}).$$

(a) Use Green's theorem to compute the difference between $\displaystyle\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$ and $\displaystyle\int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

(b) Use (a) to compute $\displaystyle\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$ (by computing the easier $\displaystyle\int_{\mathbf{c}_2} \mathbf{F} \cdot ds$, of course...).

4. QUESTIONS ABOUT DIFFERENTIAL FORMS.

(a) Let $f = \sin(xy) - 3xy^2z$. Compute the 1-form $df$.

It turns out that doing $d$ twice in a row always results in 0. The reason is a combination of "mixed partials commute", and the rules for differentials ("swapping any two $dx_i$, $dx_j$ changes the sign", and "any repeated $dx_i$ means that the form is zero").

(b) Let $\alpha$ be your answer from (a). Check the claim above by computing the 2-form $d\alpha$. I.e., compute $d\alpha$ and check that it is zero, showing the details.

The $d$ operators fit together to give a diagram

$$\left\{\begin{matrix}\text{Functions} \\ \mathbb{R}^3 \longrightarrow \mathbb{R}\end{matrix}\right\} \xrightarrow{d} \left\{\text{1-forms on } \mathbb{R}^3\right\} \xrightarrow{d} \left\{\text{2-forms on } \mathbb{R}^3\right\} \xrightarrow{d} \left\{\text{3-forms on } \mathbb{R}^3\right\},$$

where any two in a row is zero.

You have a friend who is does not like differential forms. Fortunately you have a great idea on how to help them. You think : "A 1-form is something that looks like $F_1\,dx + F_2\,dy + F_3\,dz$, where $F_1$, $F_2$, and $F_3$ are functions on $\mathbb{R}^3$. Another way to keep track of three different functions is a vector field $\mathbf{F} = (F_1, F_2, F_3)$ with three different components. I'll tell my friend to forget about 1-forms, and explain everything using vector fields."

For instance, given a function $f$, you know that $df = f_x\,dx + f_y\,dy + f_z\,dz$. In terms of your (vector field) $\leftrightarrow$ (1-form) dictionary, this is the vector field $(f_x, f_y, f_z)$. You then tell your friend : "Don't worry about the $d$ operator that takes functions to 1-forms. If you have a function $f$, $d$ of that is just the vector field $(f_x, f_y, f_z)$".

Encouraged by your success with 1-forms, you decide to simplify 2-forms and 3-forms too. A 2-form is also given by three functions, the entries in front of $dx \wedge dy$, $dy \wedge dz$, and $dx \wedge dz$. You decide to match this up with a vector field by declaring that the vector field $\mathbf{G} = (G_1, G_2, G_3)$ corresponds to the 2-form $G_1\,dy \wedge dz - G_2\,dx \wedge dz + G_3\,dx \wedge dy$. [This choice of signs will make more sense after (c) and (d) below.] You also realize that 3-forms are just multiples of $dx \wedge dy \wedge dz$, and so can be described by a single function (e.g., the 3-form $H\,dx \wedge dy \wedge dz$ would correspond to the function $H$).

(c) Using your dictionary, tell your friend how the $d$ operator takes 2-forms to 3-forms. I.e., starting with $\mathbf{G} = (G_1, G_2, G_3)$ convert $\mathbf{G}$ to a 2-form by the rule above, apply $d$ to that to get a 3-form, and then convert the 3-form back to a function (showing the details of your calculations). What function (in terms of $G_1$, $G_2$, and $G_3$) do you get?

(d) Finally, your toughest challenge : explain to your friend how to take $d$ of a 1-form using your dictionary. I.e., start with a vector field $\mathbf{F} = (F_1, F_2, F_3)$, convert it to a 1-form, take $d$ of that 1-form to get a 2-form, and then convert the 2-form back to a vector field $\mathbf{G}$ (all using the rules above). Starting with $\mathbf{F}$, what is the formula for the vector field $\mathbf{G}$ that results?

This homework assignment is due on or before **Thursday, December 7th, at 4pm**. The homework should be handed in to the mailbox of either Mike Roth (507 Jeffery Hall), or Kexue Zhang (419 Jeffery Hall).