DUE DATE: SEPT. 16, 2019

1.

(a) Use L'Hôpital's rule to show that for any integer $n \geqslant 0$, $\lim\limits_{u\to\infty} \frac{u^n}{e^u} = 0$.

(b) Use a substitution and the result of (a) to prove that $\lim\limits_{y\to\infty} \frac{y^n}{e^{y^2}} = 0$ for any integer $n > 0$.

Note that you may have to make a small extra argument when n is odd.

(c) Use a substitution and the result of (b) to prove the lemma from the first day of class: For any integer $n \geqslant 0$, $\lim\limits_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$.

2. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

In class we showed that f was three times differentiable, and using the lemma, proved the formulas

$$f'(x) = \begin{cases} \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad f''(x) = \begin{cases} \frac{4-6x^2}{x^6} \cdot e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Find $f'''(x)$ when $x \neq 0$.

(b) Find $f'''(0)$, explaining your steps, and explaining how you use the lemma.

(c) Write a piecewise formula for $f'''(x)$, similar to the formulas for f, f', and f'' above.

Now let us prove that $f(x)$ is infinitely differentiable at $x_0 = 0$. We will prove this by induction. To make the induction work, we will need to prove something stronger. We will need to prove :

($*$)

> For each $k \geqslant 0$, there is a polynomial $p_k(x)$ so that
>
> $$f^{(k)}(x) = \begin{cases} \frac{p_k(x)}{x^{3k}} \cdot e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
>
> In particular, f is has a k-th derivative at $x_0 = 0$, equal to 0.

We have already checked that statement ($*$) is true when $k = 0$, 1, 2, and 3.

(*d*) What are the polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$, and $p_3(x)$?

(*e*) Assuming that $(*)$ is true for $k = n$, use the product and chain rules to show that for $x \neq 0$, $f^{(n+1)}(x)$ is of the form $\frac{p_{n+1}(x)}{x^{3n+3}} \cdot e^{-\frac{1}{x^2}}$ for some polynomial $p_{n+1}(x)$. You do not have to find a formula for the polynomial, just show that $f^{(n+1)}(x)$ is of that form.

(*f*) Still assuming that $(*)$ is true when $k = n$, show that $f^{(n+1)}(0)$ exists, and is equal to 0. Explain carefully how you use the lemma and the inductive hypothesis in this step of the argument.

Thus, by (e) and (f), there is a polynomial $p_{n+1}(x)$ so that

$$f^{(n+1)}(x) = \begin{cases} \frac{p_{n+1}(x)}{x^{3n}} \cdot e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

proving the inductive step.

3. Write each of the following complex numbers in the form $a + bi$.

(a) $(2 + 3i)(3 + 4i)$, (b) $\dfrac{2 + 3i}{3 + 4i}$, (c) $\dfrac{1}{1 - i} + \dfrac{1}{1 + 2i} + \dfrac{1}{3 + i}$.

(d) Let $a = 3 + 4i$ and $b = 6 - 7i$. Find the real and imaginary parts of $a + bi$.

4. One of our descriptions of the complex numbers was as a vector space over $\mathbb{R}$, with basis 1, i. Fix a complex number $w = a+bi \in \mathbb{C}$. Let $\varphi\colon \mathbb{C} \longrightarrow \mathbb{C}$ be the map $\varphi(z) = wz$ (i.e., "multiplication by w").

(*a*) Show that φ is an $\mathbb{R}$-linear map.

(*b*) Write down the 2×2 matrix for φ with respect to the basis 1, i.

(*c*) What does φ do geometrically? [SUGGESTION: What is the geometric rule for multiplication?]

(*d*) Suppose that $w = -1 + 0 \cdot i = -1$. What geometrically does φ do in this case?

(*e*) Again with φ as in part (d), suppose you wanted to find a linear transformation ψ so that $\psi \circ \psi = \varphi$. What, geometrically, would ψ have to do? Can you find such a ψ?

(*f*) Does it seem strange that -1 has a square root? (Part (e) is relevant.)