1. In this question we will prove that Möbius transformations take lines and circles in $\mathbb{C}$ to lines and circles. (That is : the image of a line may be a line or a circle, and the image of a circle may be a line or a circle.) Recall that in class we have shown that for any $s \in \mathbb{R}$, $s > 0$, and any distinct w_1, $w_2 \in \mathbb{C}$, the set

$$S = \left\{ z \in \mathbb{C} \,\middle|\, |z - w_1| = s|z - w_2| \right\} \tag{1.1}$$

is a line if $s = 1$, and a circle if $s \neq 1$.

Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a function, and $S \subset \mathbb{C}$ a subset. By the notation $f(S)$ we mean

$$f(S) = \left\{ z \in \mathbb{C} \,\middle|\, \text{there exists } w \in S \text{ with } f(w) = z \right\},$$

i.e., the result of putting each element of S into f.

(a) Suppose that f is an invertible function. Use the definition above to show that

$$f(S) = \left\{ z \in \mathbb{C} \,\middle|\, f^{-1}(z) \in S \right\}.$$

Now suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an invertible matrix, with $a, b, c, d \in \mathbb{C}$, and let $\mathrm{M}_A\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the associated Möbius transformation.

(b) What is the formula for $\mathrm{M}_A^{-1}(z)$?

(c) Use the formula from (b), the result of (a), and equation (1.1) to write an equation describing those z in $\mathrm{M}_A(S)$.

(d) Rewrite the formula in (c) until you end up with an equation of the form

$$|z - u_1| = t|z - u_2|,$$

with u_1, $u_2 \in \mathbb{C}$, and $t \in \mathbb{R}$, $t > 0$. In particular, give formulas for u_1, u_2, and t in terms of w_1, w_2, s, and a, b, c, and d. (SUGGESTION : a good first step might be to multiply both sides of your answer from (c) by $|-cz + a|$.)

Thus, combining these steps, starting with a set S given by (1.1), we see that

$$\mathrm{M}_A(S) = \left\{ z \in \mathbb{C} \,\middle|\, |z - u_1| = t|z - u_2| \right\}.$$

By our result from class, the solutions to such an equation is either a line or a circle.

2. In this problem we will check the statements about lines and circles in an example. Let

$$S = \left\{ z \in \mathbb{C} \,\middle|\, |z + 2i| = 2|z - i| \right\}.$$

By the result from class, the set S is a circle in $\mathbb{C}$.

(a) Find the radius and centre of S.

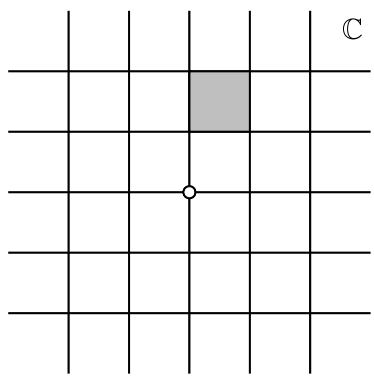
(b) Pick a point on the circle that you found in (a) and check that it satisfies the equation for S.

(c) Find the image of S under the map $\mathrm{M}(z) = \frac{1}{z}$, by finding an equation of the type "$|z - u_1| = t|z - u_2|$" describing the image.

(d) Is $\mathrm{M}(S)$ a line or a circle in $\mathbb{C}$?

3. To visualize a complex function it would be convenient if we could draw the graph. Since the graph is a subset of $\mathbb{C}^2$ (which has four real dimensions) this is not possible. The next best thing is to draw some shapes in $\mathbb{C}$ and try and visualize a function f by seeing what happens to those shapes when we apply f.

At right is a picture of the lines $\mathrm{Re}(z) \in \{-2, -1, 0, 1, 2\}$ and $\mathrm{Im}(z) \in \{-2, -1, 0, 1, 2\}$ in $\mathbb{C}$, with with the rectangle $\{z \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(z) \leqslant 1, 1 \leqslant \mathrm{Im}(z) \leqslant 2\}$ shaded in.



Draw what happens to this picture under the maps

(a) $f(z) = (1 + \sqrt{3}i)z$.

(b) $f(z) = (z - 1)^2$,

(c) $f(z) = \frac{1}{z-1}$.

Notes: (1) In (a), don't forget the geometric rule for multiplication. (2) The function in (c) is a Möbius transformation.

4. Describe (and draw) the images of the following sets under the map $f(z) = \exp(z)$.

(a) $S = \left\{ z \mid \frac{\pi}{4} \leqslant \mathrm{Im}(z) \leqslant \frac{2\pi}{3} \right\}$ (b) $S = \{ z \mid 1 \leqslant \mathrm{Re}(z) \leqslant 2 \}$.

5. Describe (and draw) the images of the following sets under the map $f(z) = \mathrm{Log}(z)$.

(a) $S = \{ z \mid \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0 \}$ (b) $S = \{ z \mid |z| \geqslant e \}$.

Note: The function Log is the principal branch of log, based on the principal branch of arg (i.e, not the multivalued log). Note that the set in (b) has a boundary.