1. Find all values of

(a) 7^i, (b) $(\sqrt{3}+i)^{\sqrt{2}}$, (c) $(1+i)^{\frac{5}{7}}$.

Next,

(d) For complex numbers z and w, is there an upper bound on $|z^w|$ in terms of $|z|$ and $|w|$? Justify your answer.

2. In this question we will prove the chain rule for complex differentiable functions. Suppose that $f\colon \mathrm{S}_1 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ and $g\colon \mathrm{S}_2 \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ are functions, with $f(\mathrm{S}_1) \subseteq \mathrm{S}_1$. (This last condition is so that the composition $g \circ f$ makes sense.)

Suppose that $z_0 \in \mathrm{S}_1$, and set $w_0 = f(z_0)$. Finally, we suppose that that f is (complex) differentiable at z_0 and that g is (complex) differentiable at w_0. We want to show :

- $g \circ f$ is differentiable at z_0; and
- the derivative at z_0 is $f'(g(z_0)) \cdot f'(z_0)$.

By the definition of the derivative, that means we need to prove that

$$\lim_{z\to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = g'(f(z_0)) \cdot f'(z_0).$$

(This statement includes the facts that the limit exists, and that it is equal to $g'(f(z_0)) \cdot f'(z_0)$.) The idea of the proof is to write this limit as

$$\lim_{z\to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z\to z_0} \left(\frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)}\right) \cdot \left(\frac{f(z) - f(z_0)}{z - z_0}\right).$$

As $z \to z_0$ the second factor on the right hand side goes to $f'(z_0)$. And, since $f(z) \to f(z_0)$ as $z \to z_0$, surely the first factor goes to $g'(f(z_0))$.

To make this precise, we introduce the following function on S_2 :

$$h(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & \text{if } w \neq w_0 \\ 0 & \text{if } w = w_0 \end{cases}$$

(a) Write down what it means (i.e, the limit definition) for "g to be differentiable at w_0 with derivative the number $g'(w_0)$".

(b) Show that h is continuous at w_0. (SUGGESTION: part (a) is relevant.)

From MATH/MTHE 281 we know that if a function is continuous at a point, then we can exchange limits and composition. In particular, since h is continuous at w_0, and since $\lim_{z\to z_0} f(z) = f(z_0) = w_0$, we know that

$$\lim_{z\to z_0} h(f(z)) = h(f(z_0)) = h(w_0) = 0.$$

(c) Give a brief justification of each of the equalities in the above equation. (E.g., which one depends on the continuity of h?)

(d) Show that (when $z \neq z_0$) :

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \Big(h(f(z)) + g'(w_0)\Big) \cdot \left(\frac{f(z) - f(z_0)}{z - z_0}\right).$$

(e) By taking the limit in (d), show that

$$\lim_{z\to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = g'(f(z_0)) \cdot f'(z_0),$$

proving the chain rule.

3. For each of the following functions, describe the domain where they are defined and compute their derivatives. (You can use the derivative rules – there is no need to differentiate the functions using the limit definition.)

(a) $f(z) = z^3 - ze^{z^2} - 4$.

(b) $g(z) = \left(z + \frac{1}{z}\right)^{10}$.

(c) $h(z) = \dfrac{z^2}{e^z - 2}$.

(d) $\mathrm{M}(z) = \dfrac{az + b}{cz + d}$ (with $ad - bc \neq 0$).