due date: Oct. 7, 2019

1. For each of the following functions $u(x, y)$ and $v(x, y)$, check if the Cauchy-Riemann equations hold. If they do hold, find a function $f(z)$ so that $f(x+iy) = u(x, y)+i\,v(x, y)$. (Your formula for f should be written in terms of z alone.)

 (a) $u(x, y) = y$, $v(x, y) = -x$.

 (b) $u(x, y) = 5x - 3y$, $v(x, y) = 3x + 5y$.

 (c) $u(x, y) = \cos(x)\sin(y)$, $v(x, y) = \sin(x)\cos(y)$.

 (d) $u(x, y) = x^3 - 3xy^2 + 1$, $v(x, y) = 3x^2y - y^3$.

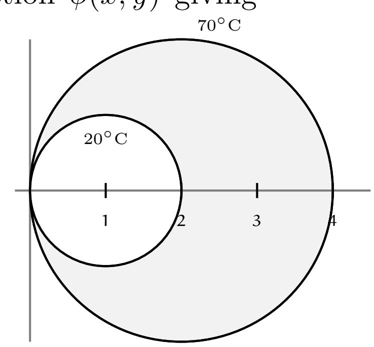
2. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Define a new function $g(z)$ by $g(z) = \overline{f(\overline{z})}$.

(i.e., given a complex number z, to compute $g(z)$ we input the complex number $\overline{z}$ into f, and then take the complex conjugate of the answer.)

Is g also an entire function? Prove this or give a counterexample.

(Suggestion: Let $p(x, y)$ and $q(x, y)$ be the real and imaginary parts of f, and use the definition of g to write its real and imaginary parts in terms of p and q. You can then investigate if the Cauchy-Riemann equations hold for g.)

3. In this problem we will use complex analysis to find the function $\phi(x, y)$ giving the steady state temperature for the shaded region shown at right. The boundary of the region consists of the circles $(x-1)^2+y^2 = 1$ and $(x-2)^2+y^2 = 4$. We suppose that the inner circle is held at a temperature of 20°C, and the outer circle at a temperature of 70°C. We are looking for the (unique, as it turns out) function $\phi(x, y)$ with the values 20 and 70 on the boundary, and harmonic in the interior. (Do not worry about what happens at the origin.)



 (a) Consider the Möbius transformation $\mathrm{M}(z) = \frac{4}{z}$. Parameterize the inner circle and (by composing your parameterization with $\mathrm{M}(z)$) show that $\mathrm{M}(z)$ takes the circle to the line $\mathrm{Re}(z) = 2$.

 (b) Similarly show that $\mathrm{M}(z)$ takes the outer circle to the line $\mathrm{Re}(z) = 1$.

 (c) Find a holomorphic function f such whose real part is equal to 70 on the line $\mathrm{Re}(z) = 1$, and whose real part is equal to 20 on the line $\mathrm{Re}(z) = 2$. (Suggestion: Perhaps a linear function would work.)

(d) Explain why $(f \circ \mathrm{M})$ is a holomorphic function whose real part is equal to 70 on the outer circle and equal to 20 on the inner circle. (You will need to mention the chain rule in your justification.)

(e) Compute $f \circ \mathrm{M}$, and find its real part ϕ.

(f) Verify that ϕ is harmonic, and check its values at the points $(2, 0)$ and $(4, 0)$.

4. For each of the following harmonic functions $u(x, y)$, find a harmonic conjugate v.

(a)$u(x, y) = \sinh(x)\sin(y) + 6x$ (b) $u(x, y) = 12x^3y - 12xy^3 + 8xy$

5. Let $u(x, y) = \ln|\sqrt{x^2 + y^2}| = \frac{1}{2}\ln|x^2 + y^2|$. Note that the domain of definition of $u(x, y)$ is $\mathrm{S} = \mathbb{C} \setminus \{0\}$.

(a) Is S simply connected?

(b) Verify that $u(x, y)$ is a harmonic function on S.

(c) Suppose we want to find a harmonic conjugate $v(x, y)$ for u. Write down the vector field $\vec{\mathbf{G}} = (\mathrm{G}_1, \mathrm{G}_2)$ so that "v is a harmonic conjugate of u" is the same as the condition that $\mathrm{grad}(v) = \vec{\mathbf{G}}$.

(d) Sketch the vector field $\vec{\mathbf{G}}$. At a point (x_0, y_0) at distance r away from the origin, what is the length of $\vec{\mathbf{G}}(x_0, y_0)$?

(e) If v exists (and $\mathrm{grad}(v) = \vec{\mathbf{G}}$) then $\vec{\mathbf{G}}$ tells us how v changes as we move in the xy-plane. Suppose we start at a point (x_0, y_0) at distance r away from the origin, and go once counterclockwise around the circle of radius r. What (according to the gradient) must happen to the value of v as we go around the circle?

(f) Can a harmonic conjugate v of u exist on all of S?

(g) What (multivalued) complex function would we be able to make into an actual function if it did?