DUE DATE: OCT. 15, 2019

1. In class we extended the definition of sin, cos, sinh, and cosh to the complex numbers. In this problem we will determine the real and imaginary parts of $\sin(z)$. (Which, it turns out, are written in terms of the real versions of sin, cos, sinh, and cosh.)

(*a*) If $x, y \in \mathbb{R}$ show that $\sin(x+iy) = \sin(x)\cosh(y) + \cos(x)\sinh(y)i$, i.e., find the real and imaginary parts of $\sin(z)$.

(*b*) Use the answer to (a) to show that for any $z \in \mathbb{C} \setminus \mathbb{R}$ (i.e., any $z \in \mathbb{C}$ which is not a real number) we have

$$\limsup_{r\to\infty} |\sin(r\cdot z)| = \infty.$$

Here r is a positive real number, going to infinity.

Thus, the statement "$|\sin(z)| \leqslant 1$ for all $z \in \mathbb{C}$" is *not* true. This is one of the facts about sin and cos which do not remain true over the complex numbers, and as we will see later, such a bound is not true for any non-constant entire function.

SUGGESTION: For part (a) it is probably easier to start on the right side of the equality, write out the definitions of sin, cos, sinh, cosh in terms of e, and then do some algebra, with the goal of getting a function that depends only on $z = x + iy$.

2. In this question we will complete the proof that $\mathrm{Log}(z)$, the principal branch of the logarithm, is holomorphic on $S = \mathbb{C} \setminus \{\text{the non-positive real axis}\}$. (I.e., we are removing the negative real axis and zero.) The definition of $\mathrm{Log}(z)$ is

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z), \text{ for } z \in S.$$

Recall that our strategy is to cover S with open subsets such that on each subset we have a formula for Arg that we understand well enough to differentiate and check that the Cauchy-Riemann equations hold. We have checked the statement for the open set

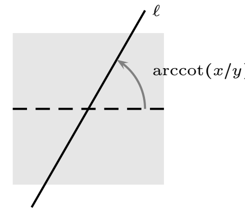
- $S_1 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, where
  $\mathrm{Log}(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\arctan(y/x)$.

In this problem we will look at the two open sets

- $S_2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, where
  $\mathrm{Log}(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\,\mathrm{arccot}(x/y)$,

and

- $S_3 = \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$, where
  $\text{Log}(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i(\text{arccot}(x/y) - \pi)$.



For a point (x, y) on a line ℓ, the function $\text{arccot}(x/y)$ returns the angle in the picture at right. The angle is always in $[0, \pi)$.

(a) To help understand how these functions fit together to give the function $\text{Arg}(z)$, fill in the table below.

$x+iy$	$\text{Arg}(x+iy)$	$\arctan(y/x)$	$\text{arccot}(x/y)$	$\text{arccot}(x/y)-\pi$	in S_1?	in S_2?	in S_3?
$1+i$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{3\pi}{4}$	✓	✓	×
$-\sqrt{3}+i$							
$1-\sqrt{3}i$							
$-1-i$							

(b) Use the identity $\frac{d}{dt}\text{arccot}(t) = -\frac{1}{1+t^2}$ and the Cauchy-Riemann theorem to show that $\text{Log}(z)$ is holomorphic on S_2 with derivative $\frac{1}{z}$.

Note: The computation in (b) also shows the result for S_3, since on S_3 the formula for the imaginary part of $\text{Log}(z)$ only differs by a constant for the formula from (b), and this constant will vanish after differentiating.

3. Let $z_0 \in \mathbb{C}$ be a fixed complex number, and for any real number $r > 0$ let γ_r be the circle of radius r around z_0, oriented counterclockwise. Find $\displaystyle\int_{\gamma_r} \frac{1}{z - z_0}\, dz$.

4. Let $f(z) = z^2$, and let $\gamma_1(t) = -1 + (t+1)i$ and $\gamma_2(t) = (t-1) + (t+1)i$. At $t = 0$, both $\gamma_1(t)$ and $\gamma_2(t)$ are at $-1+i$.

(a) Find $\gamma_1'(0)$ and $\gamma_2'(0)$.

(b) What are the lengths of each of the complex numbers in (a)? What are their angles (i.e., their arguments)?

(c) What angle do the tangent vectors in (a) (i.e., the complex numbers) make with each other, going from $\gamma_1'(0)$ to $\gamma_2'(0)$?

(d) Find $f'(-1+i)$, and compute its modulus and angle.

(e) Now compute $f(\gamma_1(t))$ and $f(\gamma_2(t))$. (I.e., put the parameterizations through f.)

(*f*) Find the derivatives of the curves you computed in (e) at $t = 0$.

(*g*) As in (b), find the lengths and angles of the complex numbers from (f).

(*h*) How have the lengths and angles changed from (b) to (g)?

(*i*) Explain the relation of your answer in (h) with your answer in (d).

(*j*) What angle to the two numbers from (f) make with each other?

(*k*) What is the relation between your answers in (c) and (j), and why does that happen?

NOTE: Because of the Thanksgiving holiday, the homework is due on **Tuesday, October 15th**.