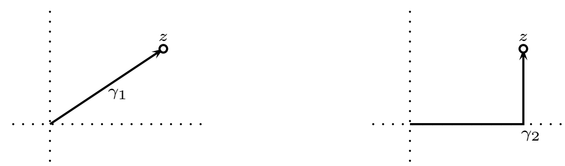
Math/Mthe 326 — Homework Assignment 6

DUE DATE: OCT. 22, 2019

1. For a point $z = x + iy \in \mathbb{C}$ let γ_1 be the path that goes in a straight line from 0 to z, and let γ_2 be the path that goes from 0 to x, and then to z, as illustrated below.



Let $f(z) = 3|z|^2$.

(a) Find $\displaystyle\int_{\gamma_1} f(z)\,dz$.

(b) Find $\displaystyle\int_{\gamma_2} f(z)\,dz$.

That is, find a formula for the integrals in (a) and (b), in terms of z (or in terms of x and y). The answers in (a) and (b) therefore define complex functions $F_1(z)$ and $F_2(z)$.

(c) Is the function F_1 from (a) holomorphic?

(d) Is the function F_2 from (b) holomorphic?

NOTE: In class when we proved the 'Fundamental theorem II', we produced a holomorphic function F by integrating a holomorphic function f along a variable path (the path, and so F depends on z). The purpose of this question is try the same thing, but with the function $3|z|^2$ instead.

2. In this problem we will practice estimating integrals.

(a) Let $\gamma_1 = \{z \in \mathbb{C} \mid |z| = 3\}$, oriented counterclockwise. Prove, that for all $z \in \gamma_1$ we have $\left|\frac{1}{z^2 - i}\right| \leqslant \frac{1}{8}$.

(b) Show that $\displaystyle\left|\int_{\gamma_1} \frac{dz}{z^2 - i}\right| \leqslant \frac{3\pi}{4}$.

(c) Let $\gamma_2 = \left\{z \in \mathbb{C} \mid |z| = e^{\pi},\ 0 \leqslant \operatorname{Arg}(z) \leqslant \frac{\pi}{2}\right\}$, oriented counterclockwise. Prove that for all $z \in \gamma_2$,

$$|\operatorname{Log}(z)| \leqslant \frac{3\pi}{2}.$$

(d) Show that $\left|\int_{\gamma_2} \operatorname{Log}(z)\,dz\right| \leqslant \frac{3\pi^2}{4} \cdot e^{\pi}$.

(e) Let γ_3 be the straight line segment from $z = 0$ to $z = 4i$. Show that for all $z \in \gamma_3$,

$$|\exp(\sin(z))| \leqslant 1.$$

(f) Show that $\left|\int_{\gamma_3} \exp(\sin z)\,dz\right| \leqslant 4$.

NOTE: For part (e), **H5**, **Q1** may be useful.

3. Let $f(z) = z^2$.

(a) By parameterizing the curve and computing the integral, find $\int_{\gamma_0} f(z)\,dz$ where γ_0 is the curve $\left\{z \mid |z| = 3, 0 \leqslant \operatorname{Arg}(z) \leqslant \frac{\pi}{2}\right\}$.

(b) By parameterizing the curve and computing the integral, find $\int_{\gamma_1} f(z)\,dz$ where γ_1 is the straight line from 3 to $3i$.

(c) By finding an antiderivative for f, calculate the integral for any path starting at 3 and ending at $3i$.

4. Let $z_0 \in \mathbb{C}$ be a fixed complex number, and for any real number $r > 0$ let γ_r be the circle of radius r around z_0, oriented counterclockwise.

(a) For which values of n is $(z - z_0)^n$ entire? When $(z - z_0)^n$ is not entire, what is its domain of definition?

(b) Find $\int_{\gamma_r} (z - z_0)^n\,dz$, where n is an integer (your answer will depend on n).

NOTE: In (b), you can use a previous homework question to take care of the case $n = -1$. For the remaining cases of (b) you should be able to use some of the theorems we know to calculate the answer without explicitly parameterizing the path and integrating the function.

Note: By popular request, and because of midterms, this homework assignment is due by 11:59pm on **Tuesday, October 22**.