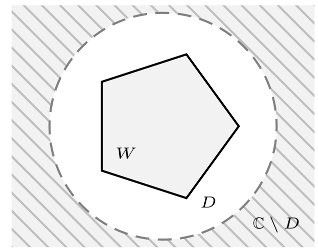
1. In this problem we will verify some of the assertions about the distance function used in the proof of Cauchy's theorem, version III (more precisely, used in the proof of the lemma that we used to prove that theorem).



Let $D \subseteq \mathbb{C}$ be a domain, and $W \subset D$ a compact set. A sketch representing some of the features of this situation is shown at right. For a point $q \in W$ we defined

$$d(q) := \min\left\{ |q-z| \,\middle|\, z \in \mathbb{C}\setminus D \right\}.$$

We want to show that the above is actually a definition (in particular, that the minimum is achieved by some point in $\mathbb{C}\setminus D$). We then want to show that the function d is continuous on W.

For a point $q \in W$, let S_q be the set $S_q = \left\{ |q-z| \,\middle|\, z \in \mathbb{C}\setminus D \right\} \subset \mathbb{R}$, i.e., the set of distances from q to a point in $\mathbb{C}\setminus D$.

(*a*) Explain why S_q is bounded from below, so that the infimum of S_q exists.

Set $e = \inf S_q$. We want to show that there is a $w \in \mathbb{C}\setminus D$ so that $e = |q-w|$. In particular, this means that $e = |q-w| \in S_q$, so that $\min S_q$ exists, and is equal to $\inf S_q$. Let z_1, z_2, z_3, ... be a sequence of points in $\mathbb{C}\setminus D$ such that $\lim_{n\to\infty} |q-z_n| = e$.

Set $\overline{D}_{e+1}(q) = \left\{ z \in \mathbb{C} \,\middle|\, |q-z| \leqslant e+1 \right\}$.

(*b*) Explain why there is an N such that for all $n \geqslant N$, $z_n \in \overline{D}_{e+1}(q)$.

(*c*) Explain why $(\mathbb{C}\setminus D) \cap \overline{D}_{e+1}(q)$ is compact.

(*d*) Explain why the sequence $\{z_n\}_{n\geqslant 1}$ has a limit point in $(\mathbb{C}\setminus D) \cap \overline{D}_{e+1}(q)$.

Let w be one of the limit points guaranteed by part (d).

(*e*) Show that $e = |q-w|$, completing the proof that $\min S_q$ exists and is equal to $\inf S_q$.

We now want to show that d is continuous on W. Fix $q_0 \in W$.

(*f*) Write out the ϵ-δ condition we have to verify in order to prove that d is continuous at q_0.

Suppose that we are given $\epsilon > 0$. Set $U = D_\epsilon(q_0) \cap W$. By part (e) there is a $w_0 \in \mathbb{C} \setminus D$ so that $d(q_0) = |q_0 - w_0|$.

(*g*) Prove that for every $q \in U$, $|q - w_0| < d(q_0) + \epsilon$. (SUGGESTION: triangle inequality.)

(*h*) Conclude from (g) that $d(q) < d(q_0) + \epsilon$.

Now fix $q_1 \in U$, and let $w_1 \in \mathbb{C} \setminus D$ be a point so that $d(q_1) = |q_1 - w_1|$ (such a point w_1 exists, again by (e)).

(*i*) Similarly to (g)+(h), show that $d(q_0) < d(q_1) + \epsilon$.

(*j*) Conclude that for all $q \in U$, $|d(q) - d(q_0)| < \epsilon$.

(*k*) Prove that d is continuous at q_0 and hence (since q_0 was arbitrary), that d is continuous on W.

2. In this problem we will prove the corollary from class :

> Let D be a simply connected domain and f a function holomorphic on D which is nowhere zero on D. Then there exists a function $g(z)$ holomorphic on D such that $f(z) = e^{g(z)}$.

The idea of the proof is that we want $g(z) =$ "$\log f(z)$". Instead of trying to define log and worrying about branch problems, if we had such a g, then we would have $g'(z) = \frac{f'(z)}{f(z)}$. This last expression is something which we can define directly from f, and then look for an antiderivative.

Consider $\frac{f'(z)}{f(z)}$. Since f is nowhere zero on D, $\frac{1}{f}$ is holomorphic on D. As we will prove soon, and independently of this corollary, if f is differentiable then f' is differentiable. Therefore f', and so $\frac{f'}{f}$ is a holomorphic function on D. Since D is simply connected a corollary of Cauchy's theorem III (from the class of Monday, October 21st) shows that there is a function $g_1(z)$, holomorphic on D, such that $g_1'(z) = \frac{f'}{f}$. We claim that, after adding the right constant c, $e^{g_1(z)+c} = f(z)$.

To see this, consider $h(z) = e^{-g_1(z)} f(z)$.

(*a*) Compute $h'(z)$ and conclude that it is zero.

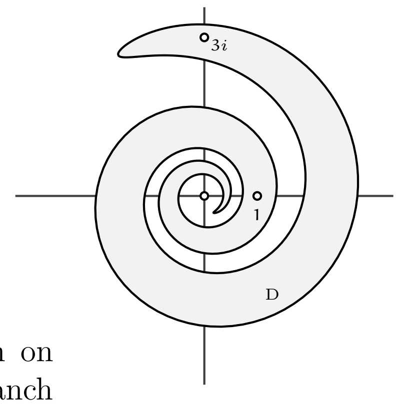
Thus, $h(z)$ is constant, equal to c_1 for some $c_1 \in \mathbb{C}$.

(*b*) Conclude that $f(z) = c_1 e^{g_1(z)}$.

(c) Explain why $c_1 \neq 0$.

Let c be any logarithm of c_1, and set $g(z) = g_1(z) + c$.

(d) Show that $e^{g(z)} = f(z)$.



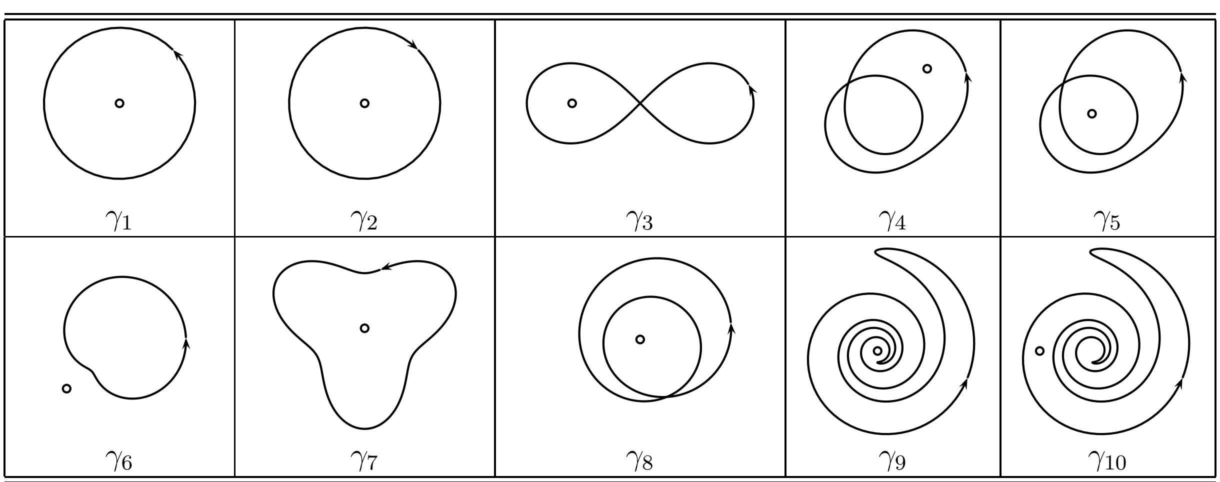
This proves the corollary.

As we have seen in class, we can apply this corollary to show that for any simply connected domain D not containing zero, there is a function $g(z)$ so that $e^{g(z)} = z$, i.e., we can define a branch of $\log z$ on D.

(e) Suppose we do this and define a branch of the logarithm on the region D shown at above right, and that for this branch $\log(1) = 0$. What is $\log(3i)$?

NOTES: (1) This corollary shows that we can define a logarithm on regions that are more interesting than just "$\mathbb{C}$ minus a half line"; (2) The real part of the logarithm is always well defined, it is the imaginary part that is more difficult. Remembering that log will be a continuous function on D, and thinking about a path from 1 to $3i$ in D, and how the angle would change when following that path, may help in answering (e).

3. In this problem we will practice thinking about the index, or winding numbers, and homotopies. Let $D = \mathbb{C} \setminus \{0\}$, and consider the following oriented loops in D (the small circle in each picture is the missing point, $z = 0$) :



(a) Find the index $\mathrm{I}(\gamma_j, 0)$ for $j = 1, \ldots, 10$.

In (a), you do not have to integrate to prove your answer, just state what you think each index is, based on looking at the loop and thinking about the physical description of the index.

(b) Which loops are homotopic to which other loops within D?

(c) Use your answer from (a) and the definition of $\mathrm{I}(\gamma_j, 0)$ to fill out the table below, giving the integrals of $\frac{1}{z}$ around each of the γ_j.

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}
$\int_\gamma \frac{1}{z}\,dz$	$2\pi i$									

In (b), give your answer by grouping loops which are homotopic to each other. E.g., "γ_a, γ_b, and γ_c are homotopic to each other, γ_d is homotopic to γ_e, γ_f is homotopic only to itself,...." In giving your answer, you do not have to write down an explicit homotopy between the loops. Instead use your geometric intuition to imagine/determine which oriented loops can be deformed continuously into which other oriented loops, all within the open set D.

Your answers for (c) should be compatible with your answers from (b) (and so also from (a)). By Cauchy's theorem, the integral of $\frac{1}{z}$ along any two loops homotopic within D will have the same value.

Note: Because of our own midterm exam this homework assignment is due by 11:59pm on **Tuesday, October 29**. Next week's homework will return to the usual Monday schedule.