Math/Mthe 326 Homework Assignment 8

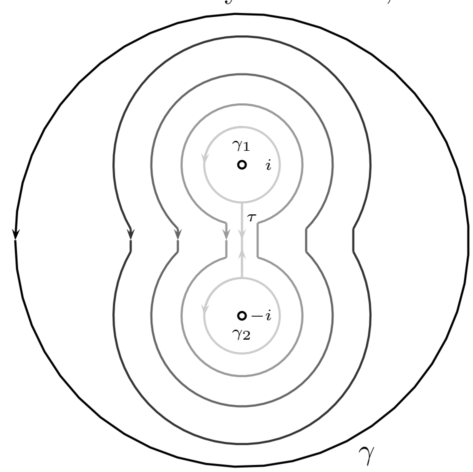
DUE DATE: Nov. 4, 2019

1. In this problem we will use path cancellation and an elementary homotopy to see more on how Cauchy's theorem helps us to compute integrals. Let $f_1(z) = \frac{2}{z-i}$, $f_2(z) = \frac{3}{z+i}$, and $f(z) = f_1(z) + f_2(z)$.

(a) What is the largest domain on which f_1 is holomorphic?

(b) What is the largest domain on which f_2 is holomorphic?

Let γ_1 be the circle of radius $\frac{1}{4}$ centered at $-i$, oriented counterclockwise, and γ_2 the circle of radius $\frac{1}{4}$ centered at i, also oriented counterclockwise. Use Cauchy's theorem, version I, or **H6**, **Q4** to answer (c)–(f) below.



(c) What is $\int_{\gamma_1} f_1(z)\,dz$?

(d) What is $\int_{\gamma_1} f_2(z)\,dz$?

(e) What is $\int_{\gamma_2} f_1(z)\,dz$?

(f) What is $\int_{\gamma_2} f_2(z)\,dz$?

Let γ be the circle of radius 3 centered at 0, oriented counterclockwise, and consider the homotopy suggested by the diagram at above right :

(g) Use Cauchy's theorem, version III, and path cancellation to explain why

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(h) Use (c)–(g) to deduce $\displaystyle\int_\gamma f(z)\,dz$.

2. Use Cauchy's integral theorem (and the version for derivatives) to solve each of the following integrals without explicitly integrating.

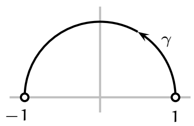
(a) $\displaystyle\frac{1}{2\pi i}\int_{|z|=1} \frac{\exp(3z)}{z^2}\,dz$ (b) $\displaystyle\int_{|z|=3} \frac{\sin(z)}{z^4} + \frac{\exp(z)}{z-2}\,dz$

(c) $\displaystyle\int_{|z|=2} \frac{1}{z(z-5)^2}\,dz$ (d) $\displaystyle\int_{|z-4|=2} \frac{1}{z(z-5)^2}\,dz$

NOTE: Leading factors (like $\frac{k!}{2\pi i}$) may have to be adjusted to make the integrals above fit the exact statement of the integral theorems. All contours should be taken counter-clockwise.

3. Let g be the function $g(w) = w^4$, γ the top half of the unit circle oriented counter-clockwise, and define the function $\mathrm{G}(z) \colon \mathbb{C} \setminus \gamma \longrightarrow \mathbb{C}$ by

$$\mathrm{G}(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - z} \, dw.$$



(a) Compute $\mathrm{G}(0)$.

(b) Compute $\mathrm{G}^{(4)}(0)$.

(c) Are the functions G and g equal on $\mathbb{C} \setminus \gamma$?

4. Let f be an entire function, $z_0 \in \mathbb{C}$ a fixed point, and γ a circle of some radius around z_0. Assuming:

(i) Cauchy's integral theorem, i.e., that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw$$

for all z inside of γ, and

(ii) that we can exchange differentiation and integration, (i.e., that we can differentiate under the integral sign),

use induction to prove Cauchy's integral formula for derivatives:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{k+1}} \, dw$$

for all z inside of γ and all $k \geqslant 0$.

5. Let n be a positive integer. We say that a function f has *polynomial growth of order at most* n if there are positive constants $a, b \in \mathbb{R}$ such that $|f(z)| \leqslant b + a|z|^n$ for all $z \in \mathbb{C}$. Show that if f is an entire function of polynomial growth of order at most n then f is a polynomial of degree at most n.

NOTE: The case $n = 0$ is Liouville's theorem, and perhaps the proof of that theorem can be modified to cover the more general version above.