1. Let $S \subseteq \mathbb{R}^2$ be a connected open set. Recall that a differentiable function $\mathrm{F}\colon S \longrightarrow \mathbb{R}^2$ is said to be *conformal* if it preserves angles and orientation. Such conformal functions, usually called conformal maps, are extremely useful in solving Dirichlet boundary problems. However, since 'preserving angles' is useful in navegation, one could also use them for constructing actual maps (e.g., of the globe). The other nice feature one may want in a physical map is that it preserves area. I.e., for any subset $S' \subset S$, the area of $F(S')$ is the same as the area of S'. In this question we will classify conformal maps which also preserve area.

The condition that a linear map $\mathrm{T}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ preserves areas is that $\det(\mathrm{T}) = \pm 1$. It follows that a one-to-one differentiable function $\mathrm{F}\colon S \longrightarrow \mathbb{R}^2$ preserves area exactly when $\det(\mathbf{D}\mathrm{F}) = \pm 1$ at each point in the domain (for instance, you could use the change of variables formula to prove this). If F is orientation preserving, the determinant of $\mathbf{D}\mathrm{F}$ must be positive, so we must have $\det(\mathbf{D}\mathrm{F}) = 1$.

In class (on Wednesday, October 2nd) we showed that a function $\mathrm{F}(x, y) = (u(x, y), v(x, y))$ is conformal if and only if the corresponding function $f(x + iy) = u(x, y) + v(x, y)i$ is holomorphic.

Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, the matrix for multiplication by the complex number $w = a + bi$.

(*a*) What is $\det(A)$?

(*b*) Express $\det(A)$ in terms of the norm of w.

(*c*) Using your answer to (b), express the condition "$\det(\mathbf{D}\mathrm{F}) = 1$", in terms of $f'(z)$.

(*d*) A recent result from class applies to the condition in (c). What is that theorem, and what does it imply about $f'(z)$?

(*e*) What does that tell us about $f(z)$? (I.e., what form must the formula for $f(z)$ have?)

(*f*) What, geometrically, does such a function $f(z)$ do?

Note that (f) then answers the question "what do area preserving conformal maps look like"?

2. In this question we will complete the proof of the theorem on differentiability of integrals of Cauchy type Let γ be a smooth compact oriented curve, and g a function continuous on γ. Recall that we define a function $\mathrm{G}\colon \mathbb{C}\setminus\gamma \longrightarrow \mathbb{C}$ by

$$\mathrm{G}(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}\,dw.$$

We want to show that G is infinitely differentiable on $\mathbb{C}\setminus\gamma$, and that the k-th derivative of G is given by the formula

$$\mathrm{G}^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{k+1}}\,dw.$$

We will prove the theorem by induction. We suppose that the theorem is true for all continuous functions on γ (even ones not called g), and all k, $0 \leqslant k \leqslant k_0$, and want to show that the theorem holds for $k = k_0+1$. Thus, for $z_0 \in \mathbb{C}\setminus\gamma$ we want to show that

- $\lim\limits_{z\to z_0} \frac{\mathrm{G}^{(k_0)}(z)-\mathrm{G}^{(k_0)}(z_0)}{z-z_0}$ exists, and so $\mathrm{G}(z)$ has a (k_0+1)-st derivative at z_0.
- This value of this limit is given by an integral formula :

$$\mathrm{G}^{(k_0+1)}(z_0) = \frac{(k_0+1)!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z_0)^{k_0+2}}\,dw.$$

In class we established the base case, $k_0 = 1$. We now fix $z_0 \in \mathbb{C}\setminus\gamma$.

(a) Show the identity $\frac{1}{(w-z)^{k_0+1}} = \frac{1}{(w-z)^{k_0}(w-z_0)} + \frac{z-z_0}{(w-z)^{k_0+1}(w-z_0)}$.

(b) Use the identity from (a) to show that

$$(*)\quad \frac{\mathrm{G}^{(k_0)}(z)-\mathrm{G}^{(k_0)}(z_0)}{z-z_0} = \underbrace{\boxed{\frac{1}{z-z_0}\cdot\left(\frac{k_0!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{k_0}(w-z_0)}\,dw - \frac{k_0!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z_0)^{k_0+1}}\,dw\right)}}_{①} + \underbrace{\boxed{\frac{k_0!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{k_0+1}(w-z_0)}\,dw}}_{②}.$$

We will deal with the limits of the two terms on the right hand side of $(*)$ separately. Term ② is easy to deal with, and the limit can be expressed by an integral :

$$\lim_{z\to z_0}\frac{k_0!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{k_0+1}(w-z_0)}\,dw = \frac{k_0!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z_0)^{k_0+2}}\,dw.$$

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To handle $\textcircled{1}$, we realize it as a difference quotient computing a k_0-th derivative, to which we can apply our induction hypothesis. For this fixed z_0, set $h(w) = \frac{g(w)}{w-z_0}$, and define the function $\mathrm{H}(z)$ as an integral of Cauchy type :

$$\mathrm{H}(z) = \frac{1}{2\pi i} \int_\gamma \frac{h(w)}{w-z}\, dw.$$

By induction the $(k_0 - 1)$-st derivative of $\mathrm{H}(z)$ is given by the formula

$$\mathrm{H}^{(k_0-1)}(z) = \frac{(k_0-1)!}{2\pi i} \int_\gamma \frac{h(w)}{(w-z)^{k_0}}\, dw.$$

(*c*) Use the formula above to express the difference quotient $\frac{\mathrm{H}^{(k_0-1)}(z)-\mathrm{H}^{(k_0-1)}(z_0)}{z-z_0}$ in terms of integrals involving h.

(*d*) Substitute the definition of h (namely $h(w) = g(w)/(w-z_0)$) into your answer from (c).

(*e*) Show that the limit we want to understand, the limit as $z \to z_0$ of $\textcircled{1}$, is k_0 times the limit of the expression in (d).

By the inductive hypothesis, $\mathrm{H}(z)$ has a k_0-th derivative at z_0, so $\lim_{z\to z_0} \frac{\mathrm{H}^{(k_0-1)}(z)-\mathrm{H}^{(k_0-1)}(z_0)}{z-z_0}$ exists. By (e), this means the limit as $z \to z_0$ of $\textcircled{1}$ exists. Thus the limits as $z \to z_0$ of $\textcircled{1}$ and $\textcircled{2}$ both exist, and hence $\mathrm{G}^{(k_0+1)}(z_0) = \lim_{z\to z_0} \frac{\mathrm{G}^{(k_0)}(z)-\mathrm{G}^{(k_0)}(z_0)}{z-z_0}$ exists.
We next need to see that this limit is equal to the integral claimed. From (e), the limit as $z \to z_0$ of $\textcircled{1}$ is equal to $k_0 \cdot \mathrm{H}^{(k_0)}(z_0)$. By induction, we know an integral formula for $\mathrm{H}^{(k_0)}(z_0)$, and so also one for $k_0 \cdot \mathrm{H}^{(k_0)}(z_0)$.

(*f*) Write out this integral formula for $k_0 \cdot \mathrm{H}^{(k_0)}(z_0)$ (in terms of h).

(*g*) Substitute the definition of h into your answer from (f).

We have now worked out integral formulas for the limits (as $z \to z_0$) of $\textcircled{1}$ and $\textcircled{2}$.

(*h*) Add these together to get an integral formula for $\mathrm{G}^{(k_0+1)}(z_0)$.

This finishes the inductive step in the proof of the theorem on differentiability of integrals of Cauchy type.

(*i*) Followup question: In class we proved the case $k_0 = 1$. The case $k_0 = 0$ is the definition of $\mathrm{G}(z)$, and so automatically true. Could we have skipped the argument in class, and just used the inductive argument above, starting with $k_0 = 0$?

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3. Is there a minimum modulus theorem?

(a) Give an example of a domain D and a holomorphic function $f(z)$ such that $|f(z)|$ achieves its minimum inside D, but $f(z)$ isn't constant. (I.e., without further hypotheses, there isn't a "minimum modulus theorem" for holomorphic functions).

(b) Suppose that D is a domain, and f a function holomorphic on D which is never zero on D. Show that if $z_0 \in D$ is a local minimum for $|f(z)|$ then f is constant near z_0.

Suggestion for (b): Consider $g(z) = \frac{1}{f(z)}$.

4. Two short questions on harmonic functions.

(a) Let $D = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. The boundary $\partial\overline{D}$ is $\{z \in \mathbb{C} \mid \operatorname{Re}(z) = 0\}$ i.e., the "y-axis", given by the equation $x = 0$. The functions $u_1(x,y) = x$ and $u_2(x,y) = 3x$ are both harmonic on D and are equal on the boundary of D. Explain why this doesn't contradict the result on the uniqueness of solutions to the Dirichlet problem. (Or, if it does contradict this result, explain what to do once mathematics collapses).

(b) Prove a Liouville-type theorem for harmonic functions: If u is function harmonic on all of $\mathbb{R}^2$, then if u is bounded above or bounded below, u must be constant.

Hint for (b): Suppose that u is a harmonic function defined on all of $\mathbb{R}^2$ and that v is its harmonic conjugate. Let $f = u + iv$ be the corresponding entire function. What is $|\exp(f(z))|$?

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