Math/Mthe 326 Homework Assignment 10

DUE DATE: NOV. 18, 2019

1. In class we used Cauchy's integral formula and Cauchy's integral formula for derivatives to deduce the existence of the Taylor series expansion for a holomorphic function $f(z)$ around a point z_0. The point of this question is to reverse that implication and show that, conversely, if we knew that each function has a Taylor expansion then we could deduce Cauchy's integral formula for derivatives (which includes Cauchy's integral formula).

Suppose that f is holomorphic in the disk $\mathrm{D}_\rho(z_0) = \left\{ z \,\middle|\, |z - z_0| < \rho \right\}$ and has Taylor series expansion

$$f(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$$

in $\mathrm{D}_\rho(z_0)$.

(a) What is the formula for the coefficients a_n in terms of the derivatives of f at z_0?

(b) Suppose that γ is a circle around z_0 of radius less than ρ (i.e., γ is contained in $D_\rho(z_0)$ so that the expansion above is valid on γ). Compute

$$\frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

by using the power series expansion for f, exchanging integration and summation, using the answer to **H6 Q4**, and the answer from (a).

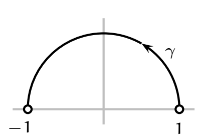
2. The Taylor series expansion

$$\frac{1}{1-z} = \sum_{k \geqslant 0} z^k = 1 + z + z^2 + z^3 + \cdots$$

valid for $|z| < 1$ is extremely useful in analysis (and combinatorics, and many other places). By differentiating this series, find the formula for the Taylor series expansion of $\frac{1}{(1-z)^n}$. You should be able to write the coefficient of z^k in the expansion as a binomial coefficient, and this is the cleanest way to write the formula.

3. Let us return to the function defined in **H8 Q3**. That is, let g be the function $g(w) = w^4$, γ the top half of the unit circle oriented counterclockwise, and define the function $\mathrm{G}(z)\colon \mathbb{C} \setminus \gamma \longrightarrow \mathbb{C}$ by

$$\mathrm{G}(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - z}\, dw.$$



As we saw in that question, $\mathrm{G}(z) \neq g(z)$. But, that leaves open the interesting question : what function *is* $\mathrm{G}(z)$?

We now have a new tool to investigate functions, namely their Taylor expansions. The Taylor expansion doesn't tell us everything about a function, but it does describe what it looks like near the centre of the expansion. In this problem we will compute the Taylor expansion of $\mathrm{G}(z)$ around $z_0 = 0$.

(*a*) Find $\dfrac{1}{2\pi i} \displaystyle\int_\gamma w^m \, dw$ for $m \in \mathbb{Z}$.

Your answer to (a) will depend on m, and you will probably need to consider the cases (i) $m = -1$; (ii) m is odd, but $m \neq -1$; and (iii) m is even, separately.

(*b*) Using your answer from (a), and the integral formula for derivatives of integrals of Cauchy type, find $\mathrm{G}^{(n)}(0)$ for all $n \geqslant 0$.

(*c*) Find the Taylor expansion of $\mathrm{G}(z)$ around $z_0 = 0$.

(*d*) What is the radius of convergence of the series from (c)?

NOTES: (1) Your answer to (c) should have an obvious pattern for the coefficients of the odd and even powers of z separately, with the exception of the coefficient for z^4. (2) In finding the radius of convergence, you may apply the ratio test to the coefficients of successive odd powers of z. (You may also use other methods of finding the radius of convergence.) (3) If there is space on a later homework, we may return to this question and see how to use this power series to give a formula for $\mathrm{G}(z)$ in terms of other functions we understand.