DUE DATE: NOV. 25, 2019

1. In this question we will use a Laurent expansion to find an infinite series for π.

(a) Find the Laurent series expansion for $\frac{1}{1+z^2}$ valid on the annulus $\mathrm{A}_{1,\infty}(0)$.

SUGGESTION: Rewrite $\frac{1}{1+z^2}$ in such a way so that you can expand part of it as a geometric series with ratio $-\frac{1}{z^2}$.

(b) By antidifferentiating the answer from (a), find a Laurent series valid on $|z| > 1$ which agrees with arctan on the positive real axis. (The only thing not determined by the integration is the constant term. However, as $z \to \infty$ on the positive real axis, you know how arctan behaves, and this will determine the constant).

(c) Use your answer from (b), the fact that $\arctan(\sqrt{3}) = \frac{\pi}{3}$ and that $\sqrt{3} > 1$ to find an infinite series for $\frac{\pi}{6}$. (The switch from $\frac{\pi}{3}$ to $\frac{\pi}{6}$ comes by combining the answer with the constant term.) Multiply this answer by 6 to find an infinite series for π.

NOTE: I know of no good use for the series expression in (c), but it is a formula for π that you probably haven't seen before.

2. Let $f(z) = \sum_{n=-\infty}^{\infty} z^n$. The purpose of this question is to show that f is the zero function in two different ways, and then show why both of these arguments are wrong.

(a) Multiply the series for f by $z-1$ and rewrite as a Laurent series (i.e., collect powers of z) to show that $(z-1)\cdot f(z) = 0$. Since $z-1$ is zero only at 1 this means that $f(z)$ must be zero at all other $z \in \mathbb{C}$. By continuity this means that f must be zero at 1 as well.

(b) Using the formula for a geometric series, show that $\frac{1}{1-z} = 1+z+z^2+\cdots$. Similarly, writing $\frac{1}{z-1}$ as $\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$, show that $\frac{1}{z-1}$ can be expanded as a series in $\frac{1}{z}$. Add these two answers to conclude again that $f(z) = 0$.

(c) The conclusions in (a) and (b) say that f is the zero function. But Laurent expansions are supposed to be unique, and surely the expansion of the zero function is just $0 = \sum_{n=-\infty}^{\infty} 0 \cdot z^n$, so this seems to be a contradiction. Explain what went wrong in parts (a) and (b).

3. Let f be a nonzero function, holomorphic on a disc $\mathrm{D}_\rho(z_0)$, and

$$f(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$$

its Taylor series expansion.

Parallel to the idea of a pole, we say that f has a *zero of order k at z_0* if $a_m = 0$ for all $m < k$, and if $a_k \neq 0$. In this problem we will establish some basic facts about this notion.

(*a*) What is the order of zero of $e^{z^3} - 1$ at $z_0 = 0$? (NOTE : e^{z^3} means $e^{(z^3)}$, not $(e^z)^3$).

(*b*) Show the following equivalent characterization of the order of zero : f has a zero of order k at z_0 if and only if $f^{(m)}(z_0) = 0$ for all $0 \leqslant m < k$, and $f^{(k)}(z_0) \neq 0$. (SUGGESTION : Use the formula for the coefficients a_n for a Taylor expansion.)

(*c*) Prove the following result, similar to our result about representing functions with a pole of order k.

> **Lemma.** Let f be a nonzero function, holomorphic in a neighbourhood of z_0. Then f has a zero of order k at z_0 if and only if f can be written as $f(z) = (z - z_0)^k g(z)$ where g is holomorphic in a neighbourhood of z_0 and $g(z_0) \neq 0$.

(*d*) Show that there is an disc $\mathrm{D}_r(z_0)$ contained in $\mathrm{D}_\rho(z_0)$ so that $f(z) \neq 0$ for all $z \in \mathrm{D}_r(z_0)$, $z \neq z_0$. I.e., show that the zeros of a nonzero holomorphic function are *isolated*. (SUGGESTION : Use (c). Since $g(z_0) \neq 0$, by continuity this must be true in some neighbourhood of z_0, and you understand the zeros of $(z - z_0)^k$.)

(*e*) Suppose that f_1 and f_2 are nonzero functions, holomorphic in a neighbourhood of z_0, with zeros at z_0 of orders k_1 and k_2 at respectively. Show that $f_1(z) \cdot f_2(z)$ has a zero of order $k_1 + k_2$ at z_0. (SUGGESTION : Use (c) or (b). Using (c) is easier.)

4. Fix $k \geqslant 1$, and let $f(z) = \frac{1}{(1-z)^k}$. Let γ be a circle of radius $\frac{1}{2}$ around 0, oriented counterclockwise. Find a formula (in n and k) for

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{n+1}}\, dw,$$

where $n \geqslant 0$.

(SUGGESTION : Combine the result of **H10 Q2** and the integral formula for a_n in a Laurent series expansion.)