DUE DATE: DEC. 2, 2019

1. Let $f(z) = \frac{5}{z+2} + \frac{3}{z-7}$. In this problem we will consider the expansion of f in different annuli around $z_0 = 0$.

(a) Find the Laurent expansion of f in the disc $\mathrm{D}_2(0) = \{z \in \mathbb{C} \mid |z| < 2\}$.

(b) Find the Laurent expansion of f in the annulus $\mathrm{A}_{2,7}(0) = \{z \in \mathbb{C} \mid 2 < |z| < 7\}$.

(c) Find the Laurent expansion of f in the annulus $\mathrm{A}_{7,\infty}(0) = \{z \in \mathbb{C} \mid 7 < |z|\}$.

NOTE: The following formulas from the class of Wednesday, November 20th will be useful in **Q1**.

- If $|z| > |\alpha|$, $\frac{1}{z-\alpha} = \frac{1}{z} \cdot \frac{1}{1-\frac{\alpha}{z}} = \frac{1}{z} \cdot \left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \frac{\alpha^3}{z^3} + \cdots\right) = \frac{1}{z} + \frac{\alpha}{z^2} + \frac{\alpha^2}{z^3} + \cdots = \sum_{n\geqslant 1} \frac{\alpha^{n-1}}{z^n}$.
- If $|z| < |\alpha|$, $\frac{1}{z-\alpha} = \frac{-1}{\alpha} \cdot \frac{1}{1-\frac{z}{\alpha}} = \frac{-1}{\alpha}\left(1 + \frac{z}{\alpha} + \frac{z^2}{\alpha^2} + \cdots\right) = -\sum_{n\geqslant 0} \frac{z^n}{\alpha^{n+1}}$.

2. For each of the following functions, find the singular points and compute the residues at those points.

(a) $\dfrac{1}{z^3(z+4)}$ (b) $\dfrac{1}{z^2+2z+1}$ (c) $\dfrac{1}{z^2-3}$

3. Suppose that f_1 and f_2 have simple poles at z_0. By writing out the Laurent expansions and multiplying, show that $f_1 f_2$ has a pole of order 2 at z_0, and find a formula for $\mathrm{Res}(f_1 f_2; z_0)$ (in terms of the coefficients of the Laurent expansions of f_1 and f_2).

4. Compute the following integrals, using the residue method. (Thus, for each problem, you will have to find the singular points, find the residues, and apply the appropriate version of the proposition for computing in the upper half plane or lower half plane.)

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)}\,dx.$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx.$

(c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x-2)(x-5)(x^2+9)}\,dx$

(d) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x-2)(x-5)(x^2+9)}\,dx$

NOTE : Since the functions in **Q4** are all real on the real line, each integral must produce a real number.

5. Compute P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{10}{z(z-i)(z-2-i)}\,dz$ using both the upper and lower half-plane versions of the propostion from the class of Wednesday, November 27th. ("Big D with hops".)

NOTE: So far all the integrals we've done were for real-valued functions, and this implies that the poles must come in conjugate pairs: If z_0 is a pole then so is $\overline{z}_0$. We're also used to seeing some correlation between the residues at the corresponding pairs of points; somehow the formulas "look the same". This example shows that this doesn't have to happen in general, and verifies (if we're allowed to use both versions), that both the upper and lower half plane methods produce the same answer, as they must, by the proposition.