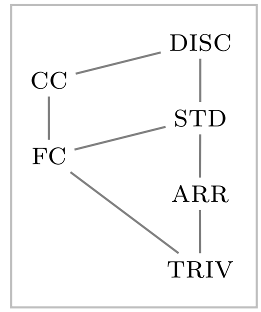
1. The Hasse diagram showing the inclusions of these topologies is shown at right, with finer topologies at the top. Your diagram might look slightly different depending on the way you decided to put the names on the page, but the connections should all be the same.



Here is a brief discussion of some of the inclusions or non-inclusions in the diagram above. TRIV is always the coarsest possible topology – every topology has to contain $\varnothing$ and the whole space, while DISC is always the finest possible topology, since in DISC every set is open. As for the others,

FC $\subseteq$ STD : Any finite set is closed in STD.

CC $\nsubseteq$ STD : Consider $\mathbb{Q} \subset \mathbb{R}$. The set $\mathbb{Q}$ is countable, so closed in CC, but is not closed in STD.

STD $\nsubseteq$ CC : The subset $[0, 1]$ is closed in STD, but is not countable, so not closed in CC.

ARR $\nsubseteq$ CC : Similarly, $(-\infty, 0]$ is closed in ARR but is not countable, so not closed in CC.

CC $\nsubseteq$ ARR : $\{0\}$ is closed in CC, but not closed in ARR. (Similarly FC $\nsubseteq$ ARR.)

ARR $\subseteq$ STD : Each (a, ∞) is also open in STD.

2.

(a) Given y_1, y_2 with $y_1 \neq y_2$, set $\epsilon = \frac{|y_2 - y_1|}{3}$, and let $V_1 = (y_1 - \epsilon, y_1 + \epsilon)$ and $V_2 = (y_2 - \epsilon, y_2 + \epsilon)$, i.e., let V_1 and V_2 be the open (1-dimensional) balls around y_1 and y_2 with radius ϵ. These sets are open in the standard topology, and have empty intersection. (Since the distance between y_1 and y_2 is 3ϵ, no point of $\mathbb{R}$ can be simultaneously within distance ϵ of y_1 and within distance ϵ of y_2, e.g., by the triangle inequality.)

(b) Let $f \colon X \longrightarrow \mathbb{R}$ be a continuous map. If f is not constant, $\mathrm{Im}(f)$ has at least two elements. Pick two such $y_1, y_2 \in \mathrm{Im}(f)$ with $y_1 \neq y_2$. By part (a) there are open sets V_1 and V_2 containing y_1 and y_2 respectively so that $V_1 \cap V_2 = \varnothing$.

Set $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then both U_1 and U_2 are nonempty (since $y_1 \in V_1$ is in $\mathrm{Im}(f)$, and similarly $y_2 \in V_2$ is in $\mathrm{Im}(f)$). But,

$$U_1 \cap U_2 = f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = f^{-1}(\varnothing) = \varnothing,$$

which, by assumption about X, cannot happen in the topology on X. Therefore f must be constant.

(c) Of the topological spaces listed above, FC, CC, ARR, and TRIV all have this property.

In ARR, we have $(a, \infty) \cap (b, \infty) = (c, \infty)$, with $c = \max(a, b)$.

Here is an argument for CC. For any two subsets U_1, U_2 of $\mathbb{R}$ we have the formula

$$\complement_{\mathbb{R}}(U_1 \cap U_2) = (\complement_{\mathbb{R}} U_1) \cup (\complement_{\mathbb{R}} U_2).$$

Thus to have $U_1 \cap U_2 = \varnothing$, it is equivalent to have

$$(\complement_{\mathbb{R}} U_1) \cup (\complement_{\mathbb{R}} U_2) = \complement_{\mathbb{R}}(\varnothing) = \mathbb{R}.$$

If U_1 and U_2 are open sets in the topology CC, then $\complement_{\mathbb{R}} U_1$ and $\complement_{\mathbb{R}} U_2$ are countable, and their union is therefore also countable, and so could not be equal to the uncountable set $\mathbb{R}$. Thus CC has the property from (b). This argument also works for FC, either by replacing "countable" with "finite", or by using the fact that FC is a coarser topology than CC, and so every open set in FC is an open set in CC, and we have just proved that nonempty open sets in CC must have nonempty intersection.

Finally, it is clear that TRIV has this property, either because it is coarser than FC (and so we can use the reasoning above), or from the simple fact that the only non-empty open set in this topology is $\mathbb{R}$ itself, and $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$, which is not empty.

3. Here is a table showing whether this particular map f is continuous in all the possible input and output topologies (at least, all among the topologies we are considering in this question), as well as the original table from the homework.

topology on the target

topology on the source

	STD	FC	CC	ARR	TRIV	DISC
STD	✓	✓	✗	✓	✓	✗
FC	✗	✓	✗	✗	✓	✗
CC	✗	✓	✓	✗	✓	✗
ARR	✗	✗	✗	✗	✓	✗
TRIV	✗	✗	✗	✗	✓	✗
DISC	✓	✓	✓	✓	✓	✓

topology on the target

	STD	FC	CC	ARR	TRIV	DISC
STD	(a)			(b)		
FC		(e)	(f)			
CC						
ARR	(c)			(d)		
TRIV					(g)	(h)
DISC					(i)	

Here are reasons for (a)–(i). There is often more than one correct argument in each case, so an argument different from the one given can still be correct.

(a) Continuous. Here there are many possible valid arguments. One possibility is that since f is a polynomial, f is continuous under the ϵ–δ definition of continuity (as seen in MATH 120!), and is therefore continuous when both sides have the standard topology.

(b) Continuous. Going from (a) to (b) we have changed the topology on the target from STD to ARR, i.e., replaced the topology on the target with a coarser one. Since f is continuous with the finer topology on the target, it is still continuous when we replace the topology on the target with a coarser one, since we are now testing a smaller collection of open sets

(c) Not continuous. The polynomial f has roots at 0 and ± 1. From the picture we therefore have $f^{-1}((0,\infty)) = (-1,0) \cup (1,\infty)$. The set $(0,\infty)$ is open in STD, but $(-1,0) \cup (1,\infty)$ is not open in ARR.

[ALTERNATE ARGUMENT : By **Q2**, a continuous function $(\mathbb{R}, \tau_{\text{ARR}}) \longrightarrow (\mathbb{R}, \tau_{\text{STD}})$ has to be constant, and f is not constant.]

(d) Not continuous. The same example above works here. $(0,\infty)$ is also open in ARR, but $f^{-1}((0,\infty))$ is not open in ARR.

(e) Continuous. Let $Z \subset \mathbb{R}$ be a closed set in the FC topology. Then Z is a finite set, say $Z = \{y_1, y_2, \ldots, y_n\}$. The set $f^{-1}(Z)$ is then the union

$$f^{-1}(Z) = \bigcup_{i=1}^{n} f^{-1}(y_i).$$

For each y_i, $f^{-1}(y_i)$ is set of solutions to the equation $f(x) = y_i$. Since f is a polynomial of degree 3, there are at most 3 solutions to each such equation. Thus, $f^{-1}(Z)$ is finite, with at most three times as many elements as Z, and therefore $f^{-1}(Z)$ is closed in FC.

(f) Not continuous. Let Z be a countably infinite set in $\mathbb{R}$ (e.g., $Z = \mathbb{Z}$), so that Z is a closed set in the topology CC. Since f is surjective, for each $y \in \mathbb{R}$ we have that $f^{-1}(y)$ is nonempty. Therefore when Z is infinite, $f^{-1}(Z)$ is infinite, and so not a closed set in FC.

(g) Continuous. Since $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ (true no matter what the map $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is), f is continuous whenever the topology on the target is TRIV.

(h) Not continuous. The set $\{0\}$ is open in DISC (every subset is open!), but $f^{-1}(\{0\}) = \{-1,0,1\}$ is not open in TRIV.

(i) Continuous. The topology on the target is TRIV and therefore the argument from (g) applies here too.

4.

(a) O_{I} : Let $U_i \in \tau_X$, $i \in I$ be a family of open subsets of X.

By definition of τ_X, for each U_i there is a $V_i \in \tau_Y$ so that $U_i = f^{-1}(V_i)$. Then

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} f^{-1}(V_i) = f^{-1}\left(\bigcup_{i\in I} V_i\right). \tag{4.1}$$

But, by O_{I} for the topology τ_Y, $\bigcup_{i\in I} V_i$ is a member of τ_Y, say $V = \bigcup_{i\in I} V_i$, $V \in \tau_Y$. Thus by definition of τ_X, $f^{-1}(V) \in \tau_X$, i.e., $\bigcup_{i\in I} U_i \in \tau_X$.

O_{II} : Let $U_i \in \tau_X$, $i \in I$ be a family of open subsets of X, with I finite. As above, for each $i \in I$ there is, by definition of τ_X, some $V_i \in \tau_Y$ with $U_i = f^{-1}(V_i)$. Also as above, we have

$$\bigcap_{i\in I} U_i = \bigcap_{i\in I} f^{-1}(V_i) = f^{-1}\left(\bigcap_{i\in I} V_i\right). \tag{4.2}$$

By O_{II} for τ_Y, $\bigcap_{i\in I} V_i \in \tau_Y$. Therefore by definition of τ_X,

$$\bigcap_{i\in I} U_i = f^{-1}\left(\bigcap_{i\in I} V_i\right) \in \tau_X.$$

O_{III} : This is clear since, for any map $f\colon X \longrightarrow Y$, $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$.

(b) Let τ'_X be any topology so that $f\colon (X, \tau'_X) \longrightarrow (Y, \tau_Y)$ is continuous. Let $U \in \tau_X$ be any element. By definition of τ_X, there is a $V \in \tau_Y$ so that $f^{-1}(V) = U$. Since $f\colon (X, \tau'_X) \longrightarrow (Y, \tau_Y)$ is continuous, this means that $U = f^{-1}(V) \in \tau'_X$. Since this is true for all $U \in \tau_X$, $\tau_X \subseteq \tau'_X$. Thus τ_X is the coarsest topology on X which makes f continuous with respect to the topology τ_Y on Y.

(c) O_{I} : Let $V_i \in \tau_Y$, $i \in I$ be a family of open sets. By definition of τ_Y, for each i, $U_i := f^{-1}(V_i) \in \tau_X$. Reading (4.1) backwards we have

$$f^{-1}\left(\bigcup_{i\in I} V_i\right) = \bigcup_{i\in I} f^{-1}(V_i) = \bigcup_{i\in I} U_i.$$

By O_{I} for τ_X, $\bigcup_{i\in I} U_i \in \tau_X$, and so by definition of τ_Y, $\bigcup_{i\in I} V_i \in \tau_Y$.

O_{II} : Let $V_i \in \tau_Y$, $i \in I$ be a family of open sets with I finite. As in (c), by definition of τ_Y for each $i \in I$ the set $U_i := f^{-1}(V_i) \in \tau_X$. Reading (4.2) in the other direction we have

$$f^{-1}\left(\bigcap_{i\in I} V_i\right) = \bigcap_{i\in I} f^{-1}(V_i) = \bigcap_{i\in I} U_i.$$

By O_{II} for τ_X, $\bigcap_{i\in I} U_i \in \tau_X$, and so by definition of τ_Y, $\bigcap_{i\in I} V_i \in \tau_Y$.

$\mathrm{O_{III}}$: The same argument as $\mathrm{O_{III}}$ in (a) works here. Since $f^{-1}(\varnothing) = \varnothing \in \tau_X$, $\varnothing \in \tau_Y$, and since $f^{-1}(Y) = X \in \tau_X$, $Y \in \tau_Y$.

(d) Let τ'_Y be any topology on Y so that $f\colon (X, \tau_X) \longrightarrow (Y, \tau'_Y)$ is continuous. Let $V \in \tau'_Y$ be any element. Since $f\colon (X, \tau_X) \longrightarrow (Y, \tau'_Y)$ is continuous, this means that $f^{-1}(V) \in \tau_X$. By definition of τ_Y, we then have $V \in \tau_Y$. Since this is true for all $V \in \tau'_Y$, we conclude that $\tau'_Y \subseteq \tau_Y$. Thus τ_Y is the finest topology on Y which makes f continuous with respect to the topology τ_X on X.