1.

(a) Here are two ways to argue that the function $p\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $p(x, y) = x$ (i.e., p is projection onto the first coordinate) is continuous, where $\mathbb{R}^2$ and $\mathbb{R}$ have the standard topologies.

(1) Continuity in the standard topologies is the same as ϵ–δ continuity, and p is a polynomial map, and so certainly ϵ–δ continuous.

(2) Or, to show continuity directly, it is sufficient to show that $p^{-1}(B_i)$ is open in $\mathbb{R}^2$, for each B_i in a generating set for the topology on $\mathbb{R}$. The usual base for the standard topology on $\mathbb{R}$ is the set of intervals $\mathcal{B} = \{(a, b) \mid a,\, b \in \mathbb{R},\, a < b\}$. For such an interval

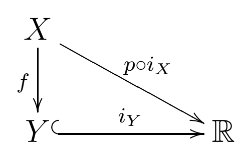
$$p^{-1}((a, b)) = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; a < x < y\right\},$$

which is an open set in the standard topology in $\mathbb{R}^2$. For instance, setting $\delta = \frac{b-a}{2}$ and $x_0 = \frac{b+a}{2}$, this set is the union of the open balls

$$\bigcup_{y \in \mathbb{R}} B_\delta(x_0, y).$$

(b) One of the properties of the subspace topology (here the subspace topology on X) is that it makes the inclusion map $i_X\colon X \hookrightarrow \mathbb{R}^2$ continuous. Therefore $p \circ i_X$, being a composition of continuous maps, is also continuous.

(c) The image of $p \circ i_X$ lies in Y. Therefore we have a factorization (at least as maps of sets) as shown at right. The f in this factorization is the map f of the question. The other property of the subspace topology (here the subspace topology on Y) is, as stated in part (c) of the theorem on properties and characterization of the subspace topology, that the map $X \longrightarrow \mathbb{R}$ being continuous implies that f is continuous. Therefore f is continuous.



(d) To show that g is continuous we need to show for every open set U in X that $g^{-1}(U)$ is open in Y, or equivalently that for every closed set W in X that $g^{-1}(W)$ is closed in Y. In general given a map $g\colon Y \longrightarrow X$ our notation "g^{-1}" does not refer to the inverse function of g, but rather the associated function $\mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$. But, if g happens to be invertible, then this function is the inverse function. (Or, more precisely, it is $\mathcal{P}_*$ of the inverse function, to use our recent notation from class.)

Thus, (in our situation where g is a bijection with inverse map f) for any subset $W \subseteq X$, $g^{-1}(W) = f(W)$. Therefore finding $W \subseteq X$ which is closed in X such that $f(W)$ is not closed in Y will show that g is not continuous.

(e) Let Z be the line $Z = \{(x,1) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$. The set Z is closed in the standard topology. Therefore $Z \cap X$ is a closed subset of X in the subspace topology.

The set of solutions to $\cos(z) = 1$ are $z \in \{2\pi k \mid k \in \mathbb{Z}\}$. The solutions z with the property that $x = \frac{1}{z} \in [-1,1]$ are those where $k \neq 0$. Therefore

$$X \cap Z = \left\{\left(\tfrac{1}{2\pi n}, 1\right) \mid n \in \mathbb{Z},\, n \neq 0\right\} = W$$

is a closed subset of X.

(f) With W as above, the set

$$f(W) = \left\{\tfrac{1}{2\pi k} \,\middle|\, k \in \mathbb{Z},\, k \neq 0\right\}$$

is not closed in Y. Let $V = \complement_Y f(W)$ be its complement. Then $0 \in V$ but there is no interval around 0 which is completely contained in V. For instance, for any small interval $(-\epsilon, \epsilon)$ (with $\epsilon > 0$) around 0, if we pick $k > \frac{1}{2\pi\epsilon}$, $k \in \mathbb{N}$, then $\frac{1}{2\pi k} \in (-\epsilon, \epsilon)$ while at the same time $\frac{1}{2\pi k} \in f(W)$, and so not in V.

Thus V is not open, and so $f(W)$ is not closed. Therefore g is not a continuous map.

2.

(a) (a1) $\overline{A}$ is the intersection of closed sets, and therefore closed.

(a2) Every set Z we are intersecting contains A (it is one of the conditions on Z). Therefore

$$A \subseteq \bigcap_{\substack{A \subseteq Z \\ Z \subseteq X \text{ closed}}} Z = \overline{A}.$$

(a3) If W is closed, and $A \subseteq W$, then W is on the list of closed subsets (the "Z"'s) we are intersecting to get $\overline{A}$. Therefore $\overline{A} \subseteq W$.

(b) Since Z is closed and contains A (i.e., (b1)+(b2)), we have by (a3) that $\overline{A} \subseteq Z$. On the other hand, $\overline{A}$ is a closed set containing A, and therefore by (b3) $Z \subseteq \overline{A}$. Together these give $Z = \overline{A}$.

(c) By (a2) $V \subseteq \overline{V}$. Since V is a closed set containing V, by (a3) we have $\overline{V} \subseteq V$. (I.e., setting $W = V$, W is a closed set containing V and so $\overline{V} \subseteq W$. Since $W = V$ this is $\overline{V} \subseteq V$.) Therefore $\overline{V} = V$.

(d) Let $V = \overline{A}$. Then V is closed, and so by (c) $\overline{V} = V$. Substituting in $V = \overline{A}$, this is $\overline{\overline{A}} = \overline{A}$.

(e) Since $\overline{A}$ and $\overline{B}$ are each closed, and the union of a finite number of closed sets is closed, $\overline{A} \cup \overline{B}$ is closed.

(f) Since $A \subset \overline{A}$ and $B \subset \overline{B}$, $A \cup B \subseteq \overline{A} \cup \overline{B}$.

(g) Since $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, by (a3) we have $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

(h) We have $A \subseteq A \cup B \overset{(a2)}{\subseteq} \overline{A \cup B}$, so $A \subseteq \overline{A \cup B}$.

(i) Since $\overline{A \cup B}$ is a closed set containing A, by (a3) we have $\overline{A} \subseteq \overline{A \cup B}$.
Symmetrically, we have $\overline{B} \subseteq \overline{A \cup B}$.

(j) Since $\overline{A}$ and $\overline{B}$ are both subsets of $\overline{A \cup B}$, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(k) Combining (g) and (j) gives $\overline{A \cup B} = \overline{A} \cup \overline{B}$.